$1.50

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W. E. Johnson

In Three Parts Part I

LOGIC

PART I

BY

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πάντα ῥεῖ εἰ μὴ τὸ ἀληθές

DOVER PUBLICATIONS, INC.
NEW YORK

Published in the United Kingdom by Constable and Company, Limited, 10 Orange Street, London W.C.2.

This Dover edition, first published in 1964, is an unabridged and unaltered republication of the work first published by the Cambridge University Press, Part I in 1921, Part II in 1922 and Part III in 1924. This edition is published by special arrangement with the Cambridge University Press.

Library of Congress Catalog Card Number 64-18362

Manufactured in the United States of America

Dover Publications, Inc.
180 Varick Street
New York 14, N.Y.

PREFACE

THE present work is intended to cover the whole field of Logic as ordinarily understood. It includes an outline of elementary Formal Logic, which should be read in close connection with Dr Keynes's classical work, in which the last word has been said on most of the fundamental problems of the subject. As regards Material Logic, I have taken Mill's *System of Logic* as the first basis of discussion, which however is subjected to important criticisms mostly on the lines of the so-called conceptualist logicians.

I have to express my great obligations to my former pupil, Miss Naomi Bentwich, without whose encouragement and valuable assistance in the composition and arrangement of the work, it would not have been produced in its present form.

W. E. J.

March 30, 1921.

'MAN IS A RATIONAL ANIMAL'

Definition

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INTRODUCTION

§ 1. Logic is most comprehensively and least controversially defined as the analysis and criticism of thought. This definition involves the least possible departure from the common understanding of the term and is not intended to restrict or extend its scope in any unusual way. The scope of logic has tended to expand in two directions—backwards into the domain of metaphysics, and forwards into that of science. These tendencies show that no rigid distinction need be drawn on the one side between logic and metaphysics, nor on the other between logic and science. The limits imposed by any writer are justified so far as his exposition exhibits unity; it is, in fact, much more important to remove confusions and errors within the subjects discussed under the head of logic, than to assign precise limits to its scope. It is, I hold, of less importance to determine the line of demarcation between logic and philosophy than that between logic and science; so that my treatment of logic might be called philosophical in comparison with that of those who implicitly or explicitly separate their criticism and analysis from what in their view should be relegated to epistemology and ontology.

This account of the scope of logic does not differ in any essential respects from that given, for example, in Mill's long introductory chapter. The special feature of Mill's logic is the great prominence given to the theory of induction, in contrast to most of his predecessors

and contemporaries, including Whately. Whately does not omit reference to induction any more than Mill omits syllogism: where they differ is that Whately asserts that in order to be valid any inductive inference must be formulated syllogistically, and that therefore the principle for induction is dependent on the principle of syllogism. Mill opposes this view; but as regards the scope of logic there is no disagreement between them: they differ simply on the question of the relations of deduction to induction.

If any writer deliberately or on principle dismisses from logic the theory of inductive inference, it must be on one of three grounds: either (*a*) that no inductive inference is valid; or (*b*) that different criteria of validity apply to different sciences; or (*c*) that the problem of the validity of induction constitutes a topic to be included in some study other than that named logic. As regards (*a*), this is the view which seems to be held by Venn in his *Empirical Logic* where, in the chapter on the subjective foundations of induction, he acknowledges that as a matter of fact human beings do make directly inductive inferences, even with a feeling of conviction, but that no warrant for such conviction can be found. Another aspect of his view of induction is expounded in the chapter on the objective foundations of induction, in which he classifies the different kinds of uniformity—such as sequence, co-existence, permanence, rhythm—which are used as major premisses, expressive of actual fact, by means of which specific uniformities under each general head are established as valid. When then he is asked what reasonable ground there is for accepting these major premisses as true, he

maintains in effect that they have to be assumed, in order to give security to the conclusions inductively inferred. In using the word assumption, there seems to be some ambiguity, namely whether it is to be understood to mean 'assumed to be true although known to be false' or 'assumed to be true although unprovable.' I take Venn to mean the latter, and that the attitude towards this assumption is merely one of felt certainty—felt, indeed, by all human beings, but having no root in our rational nature, and only exhibiting a common psychological disposition or character. This view, that there is no inductive principle that is self-evidently or demonstrably true, seems to be held by many other logicians, though none of them, I think, put it as explicitly as Venn. So while he and others include induction in their logical exposition, they neglect what I take to be the essential justification for its inclusion, namely as affording a systematic criticism of the question of its validity. As regards (*b*), many excellent text-books have been written in these days treating of the principles and methods peculiar to different sciences; it is not denied by their authors that this treatment is logical; but, if not explicitly stated, yet it seems to be suggested that in comparing the logic of one science with that of another the sole result is to exhibit differences, and that no one set of principles applies to all the different sciences. If this were the fact there would be some excuse for excluding the treatment of induction from the scope of logic, on the ground that the discussion of each of the separate principles should be relegated to its own department of science. But if, as I hold in agreement with most

other logicians, there must be a community of principle discoverable in all sciences, then the discussion of this must be included in logic. As regards (*c*) the question raised seems to be: 'Given the topic induction, what name shall be given to the science that includes it in its treatment?' rather than the converse question 'Given the name logic, shall it be defined so as to include, or so as to exclude, induction?' If we put the question in the first form, the answer is of course purely arbitrary; we might give it the name Epagogics. But if the question is put in the second form, the answer is not in the same sense arbitrary, assuming that there is general unanimity as regards the usage of the name logic to denote a science whose central or essential function is to criticise thought as valid or invalid. That induction should be included in logic thus defined follows from the undeniable fact that we do infer inductively, and that some persons in reference to some problems do infer invalidly. Even if this were not the fact, it is certainly of scientific importance to render explicit what everyone implicitly recognises in their inferences—as much for the case of induction as for that of syllogism or other formal types of inference. It has even been hinted that nobody makes mistakes in formal inference; and yet—in despite of this, if true—no one questions the value of systematising the principles under which people may unconsciously reason; and what holds of formal inference would certainly hold *à fortiori* of the processes of inductive inference which present many more serious opportunities for fallacy.

§ 2. As regards the term 'thought' which enters

into my definition, its application is intended to include perceptual judgments which are commonly contrasted with rather than subsumed under thought, for the reason that thought is conceived as purely abstract while perception contains an element of concreteness. But properly speaking even in perceptual judgment there is an element of abstraction ; and on the other hand no thought involves mere abstraction. It follows, therefore, that the processes of thinking and of perceptual judgment have an essential identity of character which justifies their treatment in a single systematic whole. It is the distinction between sense-experience and perceptual judgment, and not that between perceptual judgment and thought, that must be emphasised. The essential feature of perceptual judgment in contrast to mere sense-experience is that it involves activity, and that this activity is controlled by the purpose of attaining truth ; further it is the presence of this purpose which distinguishes thought from other forms of mental activity. Thought may therefore be defined as mental activity controlled by a single purpose, the attainment of truth.

§ 3. Now it is true, as often urged, that thought is motived not solely by the purpose of attaining truth, but rather by the intention of realising a particular end in some specific form and under certain specific circumstances. But I have to maintain that any other or further purpose which may prompt us to undertake the activity of thinking is irrelevant to the nature of thought as such, this other purpose serving only to determine the direction of activity. When such activity is actually in operation its course is wholly independent of the

prompting motive and guided by the single purpose of attaining truth. For instance, our desire for food may prompt us to search for it; but this resolve, once taken, leads to a thinking process the purpose of which is to come to some conclusion as to where food is likely to be found, and the sole aim of this process is to discover what is true on the matter in hand. This being so, the logical treatment of thought must be disencumbered from all reference to any ulterior purpose.

Whether truth is ever pursued without any ulterior purpose is a psychological question which may fairly be asked; and if introspection is to be trusted must certainly be answered in the affirmative, although the enquiry whether true knowledge has intrinsic value or not belongs to ethics. That the attainment of truth for its own sake constitutes a genuine motive force is further confirmed by recognising the fact that people do actually attach value to true knowledge, as is incontestably proved by their willingness to defy the prospect of social disapprobation, persecution, and even martyrdom incurred by the utterance and promulgation of what they hold to be true. At the same time, it must be pointed out thac the aim of the thinking process is not the attainment of truth in general, but always of truth in regard to some determinate question under consideration. This is closely analogous to the psychological fact that what we desire is never pleasure in general, but always—if the doctrine of psychological hedonism is to be accepted—some specific experience which is represented as pleasurable.

Any thinking process is normally initiated by a question and terminated by an answer; what dis-

tinguishes one thinking process from another is the difference of the question proposed. The bond of unity amongst the phases of a single process does not necessarily entail unbroken temporal continuity, but only identity of the question proposed. Indeed any thought process may be temporarily interrupted before the proposed question has been answered. It must be left as a topic for psychology to investigate the causes of such suspension, and how far the advance made serves as a starting point for further advances. Logic, on the other hand, is concerned with the nature of the advance as an advance and criticises the process from the point of view of validity or invalidity.

§ 4. The above definition of logic as the analysis and criticism of thought should be compared with that of the Scholastics, who laid emphasis on the point that logic is concerned with the art of thinking, where art is nearly equivalent to the modern term technique, and has an understood reference to activity with an end in view. The study of the art of thinking as thus understood is of use in instructing us how to proceed when thinking out any problem: for instance, it lays down rules of classification and division for the clearing up of obscurities and inconsistencies in thought; rules for the recall and selection of knowledge appropriate to any given problem; etc. Descartes' *Discourse on Method* is a classical illustration of this species of science. Modern examples of excellent treatises on these lines are to be found in Alfred Sidgwick, and other neo-pragmatists. It is a science of the highest value, and need only be separated from logic on the ground of the difference of purpose; inasmuch as its direct

purpose is the attainment of valid thought, whereas logic is the study of the conditions of valid thought, and as such it does not exclude the study of the art.

§ 5. Alongside of the use of the term 'art' to mean technique, there is a more modern usage where it implies reference to aesthetic feelings and judgments. Nowadays discussions as to whether an objective standard for these feelings and judgments should be recognised are very prominent. The nature of the feelings and judgments that enter into aesthetic appreciation belongs to psychology; but if we agree that there is a discoverable objective standard, then the treatment of the subject of aesthetics is to be distinguished from the psychological treatment, precisely as the treatment of thought in logic is distinguished from that in psychology.

Aesthetics, in this sense, raises very similar problems to those presented in Ethics; and it is frequently said that as normative Logic, Aesthetics and Ethics are related in the same way to the three psychological factors, thought, feeling and volition respectively. Each of the normative studies may be said to be based on a standard of value, the precise determination of which it is their function to formulate; in each, imperatives are laid down which are acknowledged by the individual, not on any external authority, but as self-imposed; and, in each, the ultimate appeal is to the individual's intuitive judgment. There is, however, a closer resemblance between Ethics and Aesthetics in their relations to volitions and feelings respectively, than between either of them and Logic; inasmuch as there are apparently fundamental differences of opinion

as to the ultimate ethical and aesthetical standards, that give to the studies of Ethics and Aesthetics a controversial character absent from Logic about whose standards there is no genuine disagreement. As regards the relation of Ethics to Logic, the question sometimes arises as to which subject is supreme. The answer to this question depends entirely upon the nature of the supremacy intended: the imperatives for thought become imperatives for conduct only on condition that true judgments have intrinsic value and false judgments intrinsic disvalue; and thus, from the point of view of conduct, Logic is subordinate to Ethics. On the other hand, ethical enquiry—like any other scientific investigation—has to avoid violating logical principles, so that from the point of view of true thought Logic is supreme over Ethics.

§ 6. Our discussion so far has led us to consider the relations of Logic to Philosophy in general, Psychology, Aesthetics and Ethics. Another subject to which it is closely allied and from which it is yet distinct is Grammar, the alliance being *prima facie* accounted for by the common concern of the two studies with language. The connection between thought and language presents a problem for the science of Psychology; but, so far as thinking or the communication of thought involves the use of words, the provinces of Logic and Grammar coincide; that is to say universal Grammar, which excludes what pertains to different languages and includes only what is common to all languages, should be subsumed under Logic. For the modes in which words are combined—which constitute the subject matter for Grammar—cannot be expounded or

understood except as reflecting the modes in which thoughts are combined; and this combination is effected by means of such logical operations as negation, conjunction, disjunction, alternation, implication and so on, represented by the words *not*, *and*, *not both*, *or*, *if*, etc. To justify the subordination of Grammar to Logic we have only to realise that the analysis of the sentence in Grammar corresponds to the analysis of thought in Logic, and that grammatical criticism is confined to securing that the sentence precisely represents the thought, any further criticism of the proposition coming exclusively within the province of Logic. It may be pointed out in this connection as specially significant both for the linguist and for the logician, that languages differ in the degree of their capacity to exhibit through their structure intimacy between words and thoughts.

§ 7. Amongst all the sciences over which logic must rule, there is one that occupies a unique place. The constituents of thought which are in the most narrow sense logical are those which give form to the construct, connecting alien elements by modes which give specific significance to the whole. The first group of these is expressed by ties, conjunctional words, prepositional words, and modes of verbal inflection. But as the form of thought is further elaborated there enter new kinds of terms, namely specific adjectives which have a constant meaning definable in terms of pure thought, or else are to be admitted and understood as indefinables. The most generic form of such adjectives directly expresses the result of such mental acts of comparison as like, unlike, different from, agreeing with. Owing to the purely logical nature of these relations, universal

formulae in which they are introduced can be constructed by mere abstract thought. The preliminary condition for this construction is the separating of what is given to constitute a plurality, and thus to introduce a formal factor which can only be verbally expressed by the separations and juxtapositions of the substantial words. The very general relation that separation effects is that most indeterminate relation *otherness*. When the complementary notions of separateness and togetherness are joined to constitute a unity, there enters the idea of number, and we are in the domain of mathematics.

The extraordinary capacity for development that marks mathematics is due to the precision with which the relations of comparison are capable of being amplified. Through the substitutions that are thus rendered possible, the range of application of mathematical formulae is extended beyond the bounds which would otherwise delimit logic. Any material that might be presented to thought upon which the same precise operations of comparison could be performed, would lead to the same forms as mathematics. For example ideas, not only of difference, but of determinable degrees of difference, bring the material into relations of intrinsic order, and out of these relations emanate relations between relations, so that theoretically the science develops into a highly complicated system. The point then, where we may venture to say that logic actually passes into mathematics is where the specific indefinable adjectives above referred to give new material for further logical combinations.

Here it is of great importance to point to the

relative nature of the distinction between form and matter. Logic begins with a sharp contrast between matter, as what is given as merely shapeless, and form, as that which thought imposes. But as we advance to mathematics, we impose a new element of form in introducing the relation otherness and its developments; and this being operated on by thought takes the place of new matter: in short, what is introduced as matter is form in the making. All this could be summed up by saying that for elemental logic, mathematical notions would constitute matter; whereas when the step into mathematics is once taken these same elements are just those in accordance with which thought advances in constructing more and more complicated forms. This view of the relation of logic to mathematics will be worked out in Part II of the present work under 'Demonstration,' where the procedure of building up mathematical science is shown to involve the very same principles as are used in the logical structure.

All the sciences, including mathematics, over which logic has supreme control, have been properly described as applied logic. But mathematics is applied logic in a certain very unique sense, for mathematics is nothing but an extension of logical formulae introducing none but purely logical factors; while every other science borrows its material from experiential sources, and can only use logical principles when or after such material is supplied. Within mathematics we have again the same kind of distinction, namely that between pure and applied mathematics, as it has been called. In pure mathematics, the mathematician can give free

play to his imagination in constructing forms that are restricted only by principles of logical consistency, and he develops the implications that are derivable from what may be indifferently regarded either as definitions of his fictitious constructs or as hypothetically entertained first axioms. In order that these axioms and the theorems therefrom derived may be considered as true, recourse must be had to the real world, and if applicable, the axioms come to be assertorically entertained as premisses, and the derived propositions as the developed conclusions. This application of mathematics to reality constitutes applied mathematics. Taking geometry as our first example, while there is no limit to constructing conceived spaces other than Euclidian, their application to reality demands the enquiry whether our space is or is not Euclidian. This is answered by an appeal to our immediate intuitions directed to our spatial experiences, and it is this appeal that is outside the range of pure mathematics. Again the merely logical conception of betweenness, which develops into that of serial orders of lower or higher forms of complexity, is in the first instance a product of pure logical constructiveness, and would yield implications from which a system of implicates could be developed. But such a hypothetically conceived body of propositions would have no basis in the real but for the applicability of the defined conceptions to what is given in non-mathematical intuition. This applicability holds not only in the domain of spatial order, but also in that of the qualitative relations of difference which impose serial order amongst sense impressions.

Regarded in the light of its control over all sciences

logic has been called by the name 'Methodology'; that is to say while the forms of logic implicitly control the conclusions of science, logic itself includes the study which renders explicit the ways according to which its authority is exercised. The department of logic known as methodology constitutes the third part of the present work, which is entitled 'The logical foundations of Science.'

Another illustration of applied mathematics is to quantity. Quantity is not a mere direct development from number, since a new conception, namely that of equality of units, enters as a distinctive factor which is not purely logical. It is true that equality for merely formal developments could be defined as a certain relation having the formal properties of symmetry and transitiveness, and if to this conception is added the fundamental operation plus (+), definable as a certain relation having the formal properties commutative and associative, the whole system of quantitative science could be developed without recourse to any but pure mathematical principles. But even in this range of thought quantities of different types would need recognition. For example, given the notion of length as the first spatial quantity, a new quantity is derived by multiplying length by length, which is called area; here 'multiplied' need not be more specifically defined than a certain relation having the formal properties commutative and associative. Again where a quantum of space is divided by a quantum of time, we have velocity, and in this way a totally new type of quantity is constructed and we pass from geometry to kinematics. Another quantity called mass is such that when multi-

plied by velocity there is engendered the new quantity called momentum, and when multiplied by velocity squared, energy; and in the introduction of these new species of quantity we pass from kinematics to dynamics. This is the terminus on these lines of applied mathematics; and dynamics may be defined as the science that uses the three independently definable species of quantity time, space and mass. In every extension, then, of mathematics no new idea or mode of thought need accompany the work of the calculus. It is only when the formulae have to be applied to reality, and thus to be entertained categorically, that a process of thought other than merely mathematical enters in, and intuition is directed to what is given in some form of experience. The ideas which enter into the mathematical sciences thus constructed have a form which renders them amenable to purely logical processes of indefinite degrees of complexity; this distinguishes them from the non-mathematical or 'natural' sciences that introduce ideas dependent simply upon brute matter, unamenable to logical analysis, logic entering only in the application to these ideas of classification, and the principles of inductive inference.

§ 8. Having considered logic in its relation to the different sciences, we may now pass to a discussion of its more philosophical aspects. Logicians have been classified as nominalists, conceptualists, and realists or materialists, according as they think it worth while to discuss words, thoughts or things. Names that are apt to be understood as synonyms for these have been applied to different philosophical opinions; and this fact is indicative of the change which has occurred in

the course of the history of philosophy, where the ground has been shifted from ontology to psychology, and later from psychology to logic. To take realism first. It is the name given to the Platonic view which formed the basis of Aristotle's controversy with Plato. Plato in discussing the relation between the universal and the individual, attributes *real* existence in the truest or most ultimate sense to the universal, holding that the particular individual has reality only so far as it partakes of the nature of the universal, towards which it strives as the end (ἐντελέκη) of its existence. Aristotle, opposing this view, holds that the universal exists not apart from the particular but in it.

A new psychological significance came to be attached to the term Realism, when the question of reality was raised not about the *thing*, but about the possible *idea* of the thing, these two concepts being taken to be equivalent. The so-called nominalist school of philosophers maintained the psychological view that we had no idea corresponding to a general name, along with the ontological view according to which the particular individual or concrete alone existed, and no existence could be attributed to the universal; generality, for them, attached only to names in use, and had no objective application. On the psychological point at issue the opponents of this view have been known as conceptualists, and in maintaining their opposition were led to make a psychological distinction of great importance between images and ideas. In common with the nominalists, they held that images are necessarily concrete, particular or individual, but they maintained that we can also frame ideas which can properly be

called abstract or general. Both schools assumed that images were equivalent to or at least resembled perceptions, and further that the latter were obviously concrete and particular. Berkeley represents the nominalist school, and his subtle difference from Locke—who definitely held that we can frame general ideas, though with difficulty—comes out clearly when he disputes the possibility of a general idea of a triangle (instanced by Locke) which shall be neither equilateral nor isosceles nor scalene, and from which we can in thought abstract the shape from variations of colour. In my view Locke and Berkeley were both wrong, even where they agreed; inasmuch as neither images nor perceptions reflect the concreteness and particularity of the individual thing, which should be described as determinate, in contrast to the indeterminateness of the mental processes. In fact there has been a confusion in the description of our thoughts, images and percepts, between the distinction of the universal from the particular, and that of the indeterminate from the determinate. The modern term 'generic,' which has been applied to images, should be extended also to percepts, on the ground that they share with images the character of indeterminateness—a character which must be rigidly distinguished from general or universal as properly applied to ideas or concepts.

Nominalism has yet another meaning when applied as a special logical theory; in this sense it denotes the theory according to which the proposition is an indication of the names that have been arbitrarily chosen to denote things or classes of things, and predicates merely what follows from the consistent use of these names.

Propositions are thus used as mere formulae and repeated in thought when necessary, without demanding any consideration of their meaning; so that the only ultimate foundations or premisses of knowledge are definitions, no other propositions of the nature of axioms being required. This view still clings to some modern philosophical expositions of arithmetic and pure logic, and is rather subtly akin to the view that the first premisses for science are nothing but postulates or hypotheses which, if *consistently* held, lead to the discovery of truth.

As regards Conceptualism, it is doubtful whether, as applied to the work of such writers as Hamilton and Sigwart, it can be properly regarded as a distinctive logical theory. For the prominent use of the word concept and its associate judgment points not necessarily to any difference of logical theory between those who use these words, and those who prefer the words 'term' or 'name' and 'proposition,' but merely to the common recognition that thought has form as well as verbal expression. If, however, the conceptualist proceeds to limit the scope of logic to the consideration of the forms of thought alone, then he must maintain that the truth of a judgment is tested by the form that connects the content as conceived; and conceptualism becomes equivalent to formalism. The criterion for the formalist is indeed mere consistency or coherence in fact; that for the conceptualist proper, clearness or distinctness in thought. The latter is expressed negatively by Herbert Spencer: what is clearly not conceivable is false; positively by Descartes: what is clearly conceivable is true. It follows immediately from this view that truth concerns only conceived content;

so that the direct objects of thought are not things, but our ideas about things, and judgment contains no reference to things but only to adjectives. On this understanding, the conceptualist's view is that we can only deal with things as conceived, and that it is the mode under which we conceive them that determines the adjectives themselves and their relations as constituting the content of the judgment. In this way they are led to deny all relations as subsisting between things—a denial which is simply equivalent to denying the one supreme relation *otherness*; for *otherness* may be said to be the one determinable relation to which all specific relations stand as determinates. Hence it is enough for this school of philosophy to deny the single relation otherness, and in this denial to adopt the position of monism. The view, if carried out rigidly, goes beyond that of Spinoza, who asserted that thought was *other than* extension, and even that the one Substance had an infinity of other attributes, though not conceivable by us. It is an odd fact that Lotze, in particular, explicitly rejects relations only, as expressive of the nature of Reality; but in consistency he ought to have included in his rejection ordinary adjectives. From this point of view, the only kind of singular categorical judgment concerns Reality as a whole and not any one of its several separable parts: it predicates character of the indivisible *one*, not of this or that unit in the one. Individual units, in fact, are conceived as the result of the imposition of thought to which nothing in the *one* corresponds. Thus the Monist's first principle is to deny the Pluralist's fundamental assumption that the Real, as given to thought, is given as many and as such involves existential otherness.

The conceptualist's account of the character of the singular judgment leads to a similar account of that of the particular and the universal judgment. The view is consistently borne out by his interpretation of particulars as possible conjunctions, i.e. of adjectives that we *can* conjoin in conception; and of universals as necessary conjunctions, i.e. of adjectives that we *must* conjoin in conception. Symbolically: 'Some things that are *p* are *q*' is to mean '*p* and *q* can be conjoined and can be disjoined'; 'Everything or nothing that is *p* is *q*' is to mean '*p* and *q* must be conjoined or must be disjoined.' What is true in this view is that the operations *not*, *and*, *not-both*, *if*, *or*, are supplied by thought; and that nothing in the merely objective world manifests the mere absence of a character, or the mere indeterminateness of the alternative operation, or dependence as expressed by implication. These relations are not manifested *to* thought, but analytically or synthetically discovered or rather imposed *by* thought. The view is most strikingly expressed by Mr Bradley in his dictum: only if what is possible is necessitated will it be actualised; and again, only if what is necessary is possible will it be actualised.

From conceptualism we pass back again to realism in its new sense as applying to logic, and in this application it is usually denoted by the term materialism or empiricism. We are thus led back again to Venn, and less explicitly to Mill, who contrasts the formalism or conceptualism of Hamilton with his own logical standpoint. Taking empiricism to mean that all knowledge is obtained by experience alone (as Mill only seems to have held) the doctrine amounts to maintaining that all inference is ultimately of the nature of pure induction. But taking it to mean that no knowledge gained by

experience can be validly universalised (as Venn seems to hold) then the doctrine amounts to maintaining that no inference of the nature of pure induction is valid, and that hence only deduction is guaranteed by logic. In default of any explication of which of these two views is meant by empiricism or materialism, we can only conclude that the term stands for that department of logic that is concerned with an analysis of the process of induction. But here we must note that the distinction in character between induction and deduction is not properly expressed by the antithesis of matter and form; since the relations amongst premisses and conclusion which constitute the form of an inference hold for the validity of induction as for that of deduction; and conversely, reference to the matter of the propositions is required equally for the truth of a deductive inference as for that of an inductive inference. This obvious fact has been forgotten, owing to the great prominence given by inductive logicians to the treatment of the preliminary processes of observation, search, arrangement, comparison of material data, and the formation of formulae that shall hold for the facts collected, and the aid required by experimentation. In consequence, stress is laid on the securing of correctly described premisses in the case of induction; whereas in the case of deduction stress is laid only on securing validity for the form of inference.

§ 9. In conclusion I propose to enumerate the most important features in the treatment of logical theory to be developed in the course of this work:

(*a*) The epistemic aspect of thought is included within the province of logic, and contrasted with the constitutive aspect; the former is a recognition that

knowledge depends upon the variable conditions and capacities for its acquisition; the latter refers to the content of knowledge which has in itself a logically analysable form. Such fallacies as *petitio principii* really require reference to the epistemic aspect of thought, while fallacies of the strictly formal type refer exclusively to the constitutive aspect. Again the whole theory of modality which develops into probability is essentially epistemic, indicating as it does the relation of the content of the proposition to the thinker. Thus a distinction is clearly drawn between the proposition and the attitude of assertion or judgment; and while, on this view, the proposition is identifiable when in variable relations to different thinkers, the necessity is emphasised of conceiving the proposition in terms of assertion, the act of assertion being thus taken as the complete fact to be analysed and criticised. It is this intimate connection between the assertion and the proposition which gives meaning to the identification of the adjectives true and false with the imperatives 'to be accepted' and 'to be rejected.'

(*b*) The proposition itself, which is customarily resolved into subject and predicate, is more precisely analysed by showing that the substantive alone can function as subject, and the adjective as predicate, and that these stand to one another in the relation of characterisation: the substantive being that which is characterised, the predicate that which characterises. Since an appropriate adjective can be predicated of a subject belonging to any category, including adjective, relation and proposition, the subject as thus functioning becomes a quasi-substantive. The substantive proper seems to coincide with the category 'existent,' while

if any category other than substantive stands as subject its logical nature is not thereby altered, but rather the adjectives proper to it fall under correspondingly special sub-categories determined by the category to which the subject belongs.

(*c*) Adjectives are fundamentally distinguishable into determinables and determinates, the relation between which is primarily a matter of degree, a determinable being the extreme of indeterminateness under which adjectives of different degrees of determinateness are subsumed. The relation of a determinate to its determinable resembles that of an individual to a class, but differs in some important respects. For instance, taking any given determinate, there is only one determinable to which it can belong. Moreover any one determinable is a literal *summum genus* not subsumable under any higher genus; and the absolute determinate is a literal *infima species* under which no other determinate is subsumable.

(*d*) Relations are treated as a specific kind of adjective, and are called transitive adjectives in distinction from ordinary adjectives which are intransitive. The adjectival nature of relations is apt to be obscured by the inclusion under relative terms of what are merely substantives defined by relational characterisation. All that holds universally of adjectives, including the relation of determinates to their determinable, holds of relations as such.

(*e*) Under the head of induction, fundamentally different types are distinguished. First: the very elementary process of intuitive induction, which lies at the basis of the distinction between form and matter, and by which all the formal principles of logic are estab-

lished. Next: summary induction, more usually called perfect induction, which establishes conclusions of limited universality by means of mere enumeration. Such a summary universal stands as premiss for an unlimited universal conclusion, obtained by what is called *inductio per simplicem enumerationem.* What is specially important in my treatment is the function of summary induction in the specifically geometrical form of inference. Thirdly: demonstrative induction, which employs no other principles than those which have been recognised in deduction. This species of induction is directly employed in inferring from a single experimental instance an unlimited universal; and it is this species of induction which gives the true form to the methods formulated by Mill and Bacon. Lastly we distinguish induction proper, which is conceived as essentially problematic, and as thus re-introducing the epistemic aspect in the form of probability.

(*f*) The specific notion of cause as applying to events is distinguished from the generic notion of mere determination according to a universal formula. As specific, cause relates exclusively to states or conditions temporally alterable and also referable to place; and, in this application of the notion of determination, the effect and cause are homogeneous. Not only is the character of the effect regulated by that of the cause, but the date and place of the latter is determined by the date and place of the former. The universal positional relation, as it may be called, of cause to effect is that of contiguity, which is to be conceived in the form of the coincidence of the temporal or spatial boundary of that which constitutes the cause with that which constitutes the effect. This absolute contiguity disallows any gap

between the cause process and the effect process; so that contiguity is strictly defined as equivalent to continuity. This further implies that when, as is always permissible, we conceive a phase of the causal process as temporally or spatially separated from a phase of the effect process, we must also conceive of that which goes on in the interval bridging cause and effect to be part of one continuous process. This is possible because time and space are themselves continuous. Thus change and movement are connectionally continuous, in the special sense that the character manifested at one instant of time or at one point of space differs from that manifested at another instant of time or at another point of space, in a degree the smallness of which depends upon that of the temporal or spatial interval. Again superimposed upon the continuity of this process, there is a discontinuity of the second order, ultimately due to the discontinuous occupation of space by different kinds of matter.

(*g*) The notions of cause and substance reciprocally imply one another, the latter being that which continues to exist and in which alterable states or conditions inhere. These alterable states constitute what may be called the *occurrent* or, in accordance with scholastic usage, the *occasional* causal factor. The occurrent is distinguished from and essentially connected with the *continuant* or the material factor in causation. The occurrent and continuant factors are thus united in our complete conception of substance, neither being conceivable apart from the other. This analysis gives meaning to the conception of the properties of the continuant, as potential causes which are actualised in accordance with unchanging rules by the relatively

incidental occurrences that come into being either from within or from without the continuant. In the former case the process is immanent, cause and effect being manifestations of the changeless nature of the continuant, and the temporal relation between cause and effect is here that of succession. In the latter case, the causality is transeunt, the patient being that whose state is determined, the agent being that whose alterable relation to the agent is determinative. In transeunt causality the temporal relation of cause to effect is literal simultaneity, and the critical instant at which the cause operates is that in which there is also literal geometrical contact of cause agent with effect patient. There are two fundamentally distinct types of transeunt causality. In the one case no change of state in the agent accompanies the change of state in the patient, and we have action without any direct reaction; in the other case change of state in the one directly entails change of state in the other of such a nature that the latter may be formulated as a function of the former, and here action always involves an assignable reaction. The latter case holds invariably of inter-physical causality, and again of inter-psychical causality within the sphere of a single individual's experience. But in physico-psychical causality, as also in psycho-physical causality, action never directly determines reaction, owing to the absolute disparateness between the physical and psychical in regard to the characters of the states which are predicable of the one and of the other. It is here where my treatment of logical questions transgresses into the domain of ontology; but it must be admitted that all logicians who treat these subjects inevitably transgress in the same manner.

(*h*) The position assumed by probability in logical discussion has always been dubious. On the one side the topic has been assumed to be the exclusive property of the mathematician, or rather more precisely, the arithmetician. On this view the quantity called probability is a mere abstract fraction, and the rules of probability are merely those of arithmetic. The fraction is, in short, the ratio of two numbers, the number holding for a species to that holding for its proximate genus, this ratio being necessarily a proper fraction, the limits of which are zero and unity. If this view were correct, there would be no separate topic to be called probability. A precisely reverse account of probability is that it is a measure of a certain psychological attitude of thought to which the most obvious names that could be given are belief or doubt, taken as subject to different degrees. On either of these two extreme views probability would have no particular connection with logic. The psychological account would be separated from logic, inasmuch as it would concern solely the causal explanation of different degrees of belief, and would thus give rise to no principle of rational criticism. The mere arithmetical account of probability ought in the first instance to be corrected by the recognition that the topic has its mental side. This correction requires that probability should not be expressed by a merely abstract fraction, but rather as a fraction of a certain mental quantity which may be called certainty. The psychological conditions of the variable degrees in which doubt may approximate to certainty are as such outside the province of logic; but when these various degrees are such as reason would dictate, we may speak of reasonable doubt as an assignable fraction of certitude,

thus bringing the subject into the sphere of logic. Further the quantity or degree called probability attaches exclusively to the proposition; not however to the proposition as such, but to the proposition regarded as based upon rationally certified knowledge acquired by any supposed thinker. The degree of probability is therefore referential to such knowledge, but is wholly independent of the individual thinker, being dependent solely on his rational nature, and the knowledge which he has rationally acquired.

The whole development of this aspect of the subject is to be called formal probability, and constitutes the one subject of the fourth Part of this work. The treatment of probability there developed must be distinguished from that of informal probability, that is required in discussing the foundations of science as treated in my third Part; for there, while the logic of inductive inference is made to depend upon the principles of probability and not upon any big fact about nature, yet probability is only introduced on broad and indeterminately quantitative lines. This treatment leads to an attempted enumeration of broadly formulated criteria for the evaluation of the degrees of probability to be attached to the generalisations of inductive inference. These criteria are merely expressions of what is popularly felt, and their rational justification can only be represented as depending upon postulates: that is, speculations that are neither intuitively self-evident nor experientially verifiable, but merely demanded by reason in order to supply an incentive to the endeavour to systematise the world of reality and thus give to practical action an adequate prompting motive.

CHAPTER I

THE PROPOSITION

§ 1. A SYSTEMATIC treatment of logic must begin by regarding the proposition as the unit from which the whole body of logical principles may be developed. A proposition is that of which truth and falsity can be significantly predicated. Some logicians have taken the *judgment* as their central topic, and it will be necessary to examine the distinction between what I have called a proposition and what appears to be meant by a judgment. It has been very generally held that the proposition is the *verbal expression* of the judgment; this, however, seems to be an error, because such characterisations as true or false cannot be predicated of a mere verbal expression, for which appropriate adjectives would be 'obscure,' 'ungrammatical,' 'ambiguous,' etc. There appear then to be three notions which, though intimately connected, must be clearly distinguished: namely (1) what may be called the sentence; (2) the proposition; and (3) the judgment. The sentence may be summarily defined as the verbal expression of a judgment or of a proposition; it remains, therefore, to distinguish and interrelate the proposition and the judgment.

The natural use of the term judgment is to denote an act or attitude or process which may constitute an incident in the mental history of an individual. As so conceived, we should have further to distinguish the

changing phases of a process (which might alternately involve interrogation, doubt, tentative affirmation or negation) from the terminus of such process in which a final decision replaces the variations undergone during what is commonly called suspense of judgment. It would thus be more natural to speak of passing judgment upon a proposition proposed in thought than to identify judgment as such with the proposition. This more natural usage (which is that which I shall adopt) entails the necessity of recognising the distinction between various attitudes of thought on the one hand, and the object towards which that thought may be directed on the other; and even further, when necessary, of recognising the adoption of any of these alterable attitudes of thought as a datable occurrence within the total experience of some one individual thinker. There will thus be many fundamental attributes that must be predicated of the judgment upon a proposition different from, and often diametrically opposed to, those attributes that are to be predicated of the proposition itself.

In this account the judgment is the more comprehensive or concrete term, since when seriously treated it involves the two terms thinker and proposition and, in addition, the occurrent and alterable relation that may subsist between them. In thus drawing attention to mental process in my exposition of logical doctrine, I am taking what has been unfortunately termed a 'subjective' point of view. For the term 'subjective' should be substituted 'epistemic'; and in discarding the familiar antithesis *subjective* and *objective*, it is better for the purposes of Logic to substitute the antithesis *epistemic* and *constitutive*. The epistemic side of logical doctrine points

to the quite universally acknowledged kinship of Logic with Epistemology, and, in using this term in preference to subjective, we can avoid any confusion between what belongs to Psychology as opposed to what belongs to Logic. As to the term constitutive—a term for which philosophers are indebted to Kant—it has the force of 'objective' inasmuch as it points to the constitution of such an object of thought-construction as the proposition when treated independently of this or that thinker. I may anticipate what will be treated fully in the later part of logical doctrine, by pointing out that the distinction and connection between the epistemic and constitutive sides of logical problems plays an important part in the theory of Probability; and, in my view, it ought to assume the same importance throughout the whole of the study of Logic.

Now, as regards the relation of the proposition to any such act as may be called judgment, my special contention is that the proposition cannot be usefully defined in isolation, but only in connection with some such attitude or act of thought; and I prefer to take the notion of *asserting* as central amongst these variations of attitude—which will therefore be spoken of as variations in the assertive attitude. I shall also maintain that the fundamental adjectives true and false which are (perhaps universally) predicated of mere propositions as such, derive their significance from the fact that the proposition is not so to speak a self-subsistent entity, but only a factor in the concrete act of judgment. Thus, though we may predicate of a certain proposition—say 'matter exists'—that it is true or that it is false, what this ultimately means is, that any and every thinker who might

at any time assert the proposition would be either exempt or not exempt from error. In other words, the criticism which reason may offer is directed—not to the proposition—but to the *asserting* of the proposition; and hence the customary expression that such and such a proposition is false merely means that anyone's assertion of the proposition would be erroneous. The equivalence of these two forms of criticism follows from the fundamental principle that an attitude of assertion is to be approved or condemned in total independence of the person asserting or of the time of his assertion, and in exclusive dependence upon the content of his assertion. This fundamental principle of Logic will come up for detailed treatment when the so-called Laws of Thought are explicitly discussed. In order to mark the important distinction, and at the same time the close connection, between the proposition and the act of assertion, I propose to take the term 'assertum' as a synonym for 'proposition' when such terminology may seem convenient. Thus, the assertum will coincide, not exactly with what *has been* asserted, but with what is in its nature assertible.

§ 2. Many philosophers have used the term belief in its various phases as a substitute either for judgment or for assertion; in fact, when the mental aspect of any problem assumes special prominence, the term belief as applied to the proposition is more naturally suggested than any other. While the object of belief is always a proposition, the proposition may be merely entertained in thought for future consideration, either without being believed, or in a more or less specific attitude opposed to belief, such as disbelief or doubt. To doubt

a proposition implies that we neither believe nor disbelieve it, while belief and disbelief as opposed to doubt have in common the mental characteristic of assurance. Thus there are three opposed attitudes towards a proposition, included in the distinction between assurance and doubt;—the former of which may be either (assured) belief or (assured) disbelief, and the latter of which appears further to be susceptible of varying felt degrees. The close association amongst all the terms here introduced brings into obvious prominence the mental side, which such terms as judgment or assertion seem hardly to emphasise. It would however, I think, be found that there is in reality no relevant distinction between the implications of the two terms 'judgment' and 'belief.' Those logicians who have spoken exclusively of judgment, conception, reasoning, etc., have had in view more complicated processes, the products of which have been explicitly formulated; while those who have used belief and cognate terms have included more primitive and simple processes, the products of which may not have been explicitly formulated. Since the traditional logic has treated *only* the more developed processes, the term judgment and its associates is perhaps preferable for this somewhat limited view of the scope of Logic, while the use of the term belief—which must certainly be understood to include the higher as well as the lower processes—points to a wider conception of the province of Logic. To put the matter shortly, I hold it to be of fundamental importance to insist that there is *some* factor common to the lower and higher stages, and that this common factor, to which the name belief has been given, is necessarily directed to what

in Logic is called a proposition[1]. Assertion, in the sense here adopted, is to be understood to involve belief, and may be defined as equivalent to *conscious belief*. This definition restricts the term in two ways: in that, firstly, to assert does not merely mean to *utter* (without belief); and secondly, merely to believe *unconsciously* is not to assert.

§ 3. In speaking of variations of attitude towards the proposition, an assumption is involved that there is a single entity called the proposition that is the same whatever may be the attitude adopted towards it. Ordinary language supplies us with names for such different attitudes along with cognate names for the proposition: thus we associate 'to assume' with 'an assumption'; 'to suppose' with 'a supposition'; 'to propose' with 'a proposition'; 'to postulate' with 'a postulate'; 'to presume' with 'a presumption'; etc.[2] Consider the two verbs 'to assume' and 'to presume.' It will be acknowledged that these denote attitudes between which some subtle distinction may be under-

[1] Readers of Psychology should be warned that, when psychologists contrast 'imagination' with 'belief,' each term indicates an attitude to a proposition; while, when they contrast 'imagination' with 'perception,' the processes to which they refer do not involve any attitude towards a proposition. There is no common element of meaning in these two applications of the word 'imagination.'

[2] In further illustration of this point we may select certain prominent logical terms such as hypothesis, postulate, axiom. Each of these terms indicates the peculiar attitude *to be* assumed towards the proposition in question by *any* thinker: thus a hypothesis stands for a proposition which awaits further scientific investigation before being finally accepted or rejected; a postulate stands for a proposition which cannot be brought to the test of experience, but the truth of which is *demanded* by the thinker; and an axiom is a proposition the truth of which is self-evident to the thinker.

stood; and thus it might appear that in correspondence with this distinction there must be a similar subtle distinction between an assumption and a presumption. Unfortunately substantival words such as these are apt to suggest a difference in nature between that which in the one case is presumed and in the other assumed; but this suggestion must be rejected, and it must be maintained on the contrary that the content of a proposition preserves its identity unmodified, independently of all variations of assertive attitude and of personal and temporal reference. This independence holds also in regard to what has been termed 'logical' in contrast with 'psychological' assertion. The phrase *logically asserted*, applied to this or that proposition, is only metaphorically legitimate, and literally equivalent to 'asserted on purely rational grounds by any or all rational persons.' In other words, the predicate 'asserted' conveys no meaning when taken apart from a person asserting.

Adopting as we do the general view that no logical treatment is finally sound which does not take account of the mental attitude in thought, it follows that the fundamental terms 'true' and 'false' can only derive their meaning from the point of view of criticising a certain possible mental attitude. We are thus bound to distinguish the object of this attitude (the assertum) from the attitude itself which may vary independently of the object; but we can only avoid contradiction or vagueness if, while permitting ourselves to distinguish between the attitude and its object, we at the same time refuse to separate them. We may further explain the adjectives 'true' and 'false' so as to bring out what

characterises logic in contrast with—or rather in its relation to—psychology: namely that logic formulates standards or imperatives which as such have no significance except as imposed upon mental acts. Thus we may say that the application of the adjectives true and false coincides with the application of the imperatives 'to be accepted' and 'to be rejected' respectively. We may add that these imperatives are imposed by the thinker—in the exercise of his reason—upon himself. In maintaining this coincidence between the two imperatives on the one hand and the two adjectives (true and false) on the other, it must not be taken that we are able thus to *define* the adjectives true and false. On the contrary, we are forced to insist that they are indefinable. We are only indicating that a reference to mental attitude is presupposed when Logic recognises the distinction between true and false in its formulation of standards for testing the correctness of a judgment or assertion.

§ 4. So far we have taken the proposition *as a unit* of which the adjectives true and false may be predicated. Before proceeding to *analyse* the proposition into its component parts, a word must be said in regard to the relation of logic to universal grammar, and in particular the relation between grammatical and logical analysis. Properly speaking, grammatical analysis cannot be regarded as dealing merely with words and their combinations. The understanding of the grammatical structure of a sentence—which includes such relations as those of subject to predicate, and of subordinate to co-ordinate clauses—requires us to penetrate below the mere verbal construction and to consider the formal structure of

thought. Hence, on the one hand, grammar cannot be intelligently studied unless it is treated as a department of logic; and, on the other hand, logic cannot proceed without such a preliminary account of linguistic structure as is commonly relegated to grammar. In short, universal grammar (as it is called) must be subsumed under Logic. On this view, a slight alteration in grammatical nomenclature will be required, whereby, for the usual names of the parts of speech, we substitute substantive-word or substantive-phrase, adjective-word or adjective-phrase, preposition-word or phrase, etc., reserving the terms substantive, adjective, preposition, etc., for the different kinds of entity to which the several parts of speech correspond.

§ 5. To turn now to the analysis of the proposition. We find that in every proposition we are determining *in* thought the character of an object presented *to* thought to be thus determined. In the most fundamental sense, then, we may speak of a determinandum and a determinans: the determinandum is defined as what is presented *to be* determined or characterised by thought or cognition; the determinans as what *does* characterise or determine in thought that which is given to be determined. We shall regard the substantive (used in its widest grammatical sense) as the determinandum, and the adjective as the determinans. Neither of these terms can be defined except in their relation to one another as each functions in a possible proposition. As it has frequently been said, the proposition is *par excellence* the unit of thought. This dictum means that the logical nature of any components into which we may analyse the proposition can only be defined by the mode in

which they enter into relation within it. For example, when I use *determinandum* for the substantive and *determinans* for the adjective, I am only defining the one in terms of the other, inasmuch as the common factor 'determine' is contained in both. This account goes beyond that which has become commonplace among many philosophers, namely, that the subject of a proposition is ultimately something which cannot be defined in the way in which a predicate or adjective can be defined; for to this we have to add that the predicate of a proposition is ultimately something which cannot be defined in the way in which a subject or substantive can be defined. These two statements present the natures of subject and predicate purely negatively, the positive element being supplied by the terms 'determinans' and 'determinandum.'

We have now to examine the nature of the connection involved in every case where adjective and substantive are joined; for example 'a cold sensation,' 'a tall man.' In order to understand the verbal juxtaposition of substantive and adjective, we must recognise a latent element of form in this construct, which differentiates it from other constructs—which also are necessarily expressed by a juxtaposition of words. This element of form constitutes what I shall call the *characterising tie.* The general term 'tie' is used to denote what is not a component of a construct, but is involved in understanding the specific form of unity that gives significance to the construct; and the specific term 'characterising tie' denotes what is involved in understanding the junction of substantive with adjective. The invariable verbal expression for the characterising tie is the verb

'to be' in one or other of its different modes. To think of 'a tall man' or of 'a cold sensation' is to think of 'a man as being tall,' 'a sensation as being cold.' Here the word 'being' expresses the characterising tie, and the fact that in some cases the word may be omitted is further evidence that the tie is not an additional component in the construct, but a mere formal element, indicating the connection of substantive to adjective. This is its peculiar and sole function; and, as the expression of the unique connection that subsists between substantive and adjective, it is entirely unmodifiable.

The distinction and connection between substantive and adjective correspond to—and, in my view, explain—the distinction and connection between particular and universal[1]. Ultimately a universal means an adjective that may characterise a particular, and a particular means a substantive that may be characterised by a universal. The terms particular (or substantive) and universal (or adjective) cannot be defined as functioning in isolation, but only as they enter into union with one another. There is some danger of confusing two different uses of the verb 'to characterise,' which may be partly responsible for the historical dispute concerning the relation of particular to universal. Primarily the term 'characterise' should be used to connect substantive with adjective in the form 'such and such a quality or adjective characterises such and such an object or substantive.' On the other hand, in the phrase 'the thinker characterises such or such an object,' characterises means

[1] Here the terms particular and universal are used in the sense current in philosophy, and not in their familiar application in elementary logic, where they stand for sub-divisions of the proposition.

'cognitively determines the character of.' Owing to this elliptical use of the term, the particular has been conceived of as 'an uncharacterised object,' and this would mean literally 'an object without any character'; but since actually every object must have character, the only proper meaning for the phrase 'uncharacterised object' is 'an object whose character has not been cognitively determined.' If then the term 'exist' may be predicated equally of a universal as of a particular, then we may agree with the Aristotelian dictum that the universal exists, not apart from, but *in* the particular; and by this is meant that the adjective exists, not apart from, but as characterising its substantive; to which must be added that the substantive exists, not apart from, but as characterised by its adjective. Now *in thought* the substantive and the adjective may be said to be separately and independently represented; hence thinking effects a severance between the adjective and the substantive, these being reunited in the *asserted* proposition—not only by the characterising tie, but *also* by what we may call the *assertive* tie. The blending of the assertive with the characterising tie is expressed in language by the transition from the participial, subordinate, or relative clause, to the finite or declaratory form of the principal verb. Thus in passing from 'a child fearing a dog' to 'a child fears a dog,' the characterising tie joins the same elements, in the same way, in both cases; but is, in the latter, blended with the assertive tie. That the ties are thus blended is further shown by the modifications 'is-not,' 'may be,' 'must be,' by which the verb 'to be' is inflected in order to indicate variations in the assertive attitude while the character-

ising relation remains unchanged. The copula '*is*' of traditional logic is thus seen to be a blend of the characterising with the assertive tie.

§ 6. We must now criticise a view, explicitly opposed to our own, as to the nature of the copula *is*. There has been for a long period an assumption that the proposition in some way or other asserts the relation of identity. This relation of identity, it is admitted, is not one of complete or absolute identity, but involves also a relation of difference: thus the proposition 'Socrates is mortal' is transformed into 'Socrates is a mortal being'—where 'Socrates' and 'a mortal being' are affirmed to be identical in denotation but different in connotation. Have logicians quite recognised the extreme elaborateness of this verbal transformation? The adjective 'mortal' has first to be turned into a substantive in using the word 'a mortal being'; secondly, the indefinite article has to be introduced, since it is clear that Socrates is not identical with every mortal; thirdly, the indefinite article has to be carefully defined as meaning *one or other*; fourthly, the relation of the adjective 'mortal' to the substantive 'being' which it characterises still remains to be elucidated; fifthly, another adjective (a relational adjective) namely *identical* is introduced in the compound phrase 'is identical with.' The proposition finally becomes: 'Socrates is identical with one or other being that is mortal.' Here the two adjectives 'mortal' and 'identical with' are each introduced after *is*. Now, if 'is identical with' is to be substituted for *is* in each case, then we shall arrive at an infinite regress. Thus, in the first place, 'Socrates *is* identical with X' (say) must be rendered 'Socrates

is identical with a being that is identical with X' where the force of *is* still remains unexplained. And in the second place 'one or other being that *is* mortal' must be rendered 'one or other being that *is identical with* a mortal,' where again *is* still remains to be explained. In each case, if an infinite regress is to be avoided, the word *is* that remains must be interpreted as representing the unique mode in which the fundamentally distinct categories substantive and adjective are joined.

§ 7. Having so far considered the proposition in its mental or *subjective* aspect, we have next to examine it in what may be called its *objective* aspect. Whereas a proposition is related subjectively to *assertion*, we shall find that it is related objectively to *fact*[1]. Our conclusion, briefly expressed, is that any proposition *characterises* some fact, so that the relation of proposition to fact is the same as that of adjective to substantive. Bradley has represented a proposition as ultimately an adjective characterising Reality, and Bosanquet as an adjective characterising that fragment of Reality with which we are in immediate contact. In adopting the principle that a proposition may be said, in general, to characterise a fact, I am including with some modification what is common to these two points of view.

One parallel that can be drawn between the relation of an adjective to a substantive and that of a proposition to a fact is that, corresponding to a single given substantive, there are an indefinite number of adjectives which are truly predicable of it, just as there are many different propositions which truly characterise any given

[1] Otherwise expressed: The proposition, subjectively regarded, is [redacted] *assertibile*; objectively regarded, [redacted] *possibile*.

fact. Thus we do not say that corresponding to a single fact there is a single proposition, but on the contrary, corresponding to a single fact there is an indefinite number of distinct propositions. Again, just as amongst the adjectives which can be truly predicated as characterising a given substantive, some are related to others as relatively more determinate; so, amongst the several propositions which truly characterise a single fact, some characterise it more determinately and thus imply those which characterise the same fact less determinately. We may therefore regard the process of development in thought as starting from a fact given to be characterised, and proceeding from a less to a continually more determinate characterisation.

Again there is an exact parallel between the relation of contradiction or contrariety amongst adjectives that could be predicated of a given substantive, and amongst propositions which could be formulated as characterising a given fact. Thus the impossibility of predicating certain pairs of adjectives of the *same substantive* involves the same principle as the impossibility of characterising the *same fact* by certain pairs of propositions: such pairs of adjectives and propositions are *incompatible*, and this relation of *incompatibility* lies at the root of the notion of contradiction. We may illustrate the relation of incompatibility amongst adjectives by *red* and *green* regarded as characterising the same patch. It is upon this relation of incompatibility that the idea of the contradictory *not-red* depends; for *not-red* means some adjective incompatible with *red*, and predicates indeterminately what is predicated determinately by *green*, or by *blue*, or by *yellow*, etc. Amongst propositions

the relation of incompatibility may be illustrated by 'Every p is qu' and 'Every pv is non-q,' which are more determinate forms of the pair of contradictory propositions 'Every p is q' and 'Some p is non-q.' These latter derive their significance as mutually contradictory from the principle that the actual fact must be such that it could be characterised *either* by such a relatively determinate proposition as 'Every p is qu' *or* by such a relatively determinate proposition as 'Every pv is non-q.'

This account of the relation of contradiction as ultimately derived from that of incompatibility or contrariety (whether applied to adjectives regarded as characterising substantives or to propositions regarded as characterising facts) brings out in another aspect the principle that any given substantive or any given fact may be truly characterised by a more or by a less determinate adjective or proposition: a topic which will be further developed in later chapters.

The above logical exposition of the nature of a proposition leads to a consideration of the philosophical problem of the relation of thought to reality in one of its aspects. It is at the present day agreed that this relation cannot be taken to be identity, and the notion of correspondence has been put forward in its place. The above account enables us to give a more definite exposition of what more precisely this so-called correspondence entails: the truth of a judgment (expressed in a proposition) may be said to mean that the proposition is in accordance with a certain fact, while any proposition whose falsity would necessarily follow from the truth of the former is in discordance with that fact.

In this way the somewhat vague conception of the correspondence between thought and reality is replaced by the relation of accordance with a certain fact attributed to the true proposition, and of discordance with the same fact attributed to the associated false proposition.

CHAPTER II

THE PRIMITIVE PROPOSITION

§ 1. THE form of proposition which appears to be psychologically prior even to the most elementary proposition that can be explicitly analysed is the exclamatory or impersonal. Propositions of this kind, which are more or less unformulated and which may be taken to indicate the early stages in a developing process, will here be called *primitive*. The most formless of such primitive propositions is the exclamatory assertion illustrated by such an utterance as 'Lightning!' This appears to contain only a characterising adjective without any assigned subject which is so characterised. Now it is true that any proposition can be regarded as a characterisation of the universe of reality regarded as a sort of unitary whole; but this way of conceiving the nature of the proposition in general, must be also associated with the possibility of using adjectives as characterising a part rather than merely the whole of reality; and certainly the case here is one in which we are bound to recognise the lightning as having, so to speak, an assignable place within the universe, and not merely as an adjective attached to the universe as a whole. The lightning as an actual occurrence must occupy a determinate position, in reference both to time and to space; but it is obvious that no reference to such determinate position is itself contained in the merely exclamatory assertion. Any implicit reference to place

or time can only be rendered explicit when the judgment has been further developed; in the undeveloped judgment the reference is indeterminate, and any judgment which might be developed from this primitive form would assert what was unasserted in the original. In logical analysis it is of the utmost importance to avoid putting into an assertion what further development of the percipient's thought might elicit on the basis of the original.

We ask then, how such judgment in its most primitive and undeveloped form can be conceived as referring to a subject when its verbal expression includes no such reference? Now we may speak of the presented occasions or occurrences that give rise to such incompletely formulated judgments as *manifestations of reality*. The exclamatory judgment 'Lightning' may thus be rendered formally complete by taking as subject term 'a manifestation of reality.' Here I do not propose to take simply as the equivalent of the exclamatory judgment 'Reality is being manifested in the lightning,' but rather 'A *particular portion of reality* manifests the character (indicated by the adjectival import of the word) lightning.' In short, what is asserted by the percipient is '*a* manifestation of lightning[1].' This phrase for representing the assertum contains of course the characterising tie but not the assertive tie. The assertive tie may be introduced by employing the form: '*There is* a manifestation of lightning,' which raises the interesting problem as to the significance of the word

[1] In grammatical phraseology, the expression 'manifestation of reality' illustrates the *subjective* genitive, while 'manifestation of lightning' illustrates the *objective* genitive.

'there.' Like many other words in current language it is used here in a metaphorical, or perhaps rather in a general or abstract sense. Literally 'there' means 'in that place,' so that in its original significance it involves the demonstrative article, and furthermore—which is the new matter of interest—a reference to position in space. Moreover the tense of the verb *is* points to the present time. If these references were developed still more precisely, the assertion would become: 'There and now—in that place and at this time—is a manifestation of lightning.' What remains as the significant element in the word 'there is,' in the absence of any *definite* reference to position in time or space, must be an *indefinite* reference to position in time and space. Otherwise the exclamatory assertion can only be expressed by omitting the word 'there' altogether, and the assertion to which we are reduced—when the subject implicit in the exclamation is made explicit—becomes, as above, '*A* manifestation of lightning[1].'

§ 2. The phrase 'there is' points to an important presupposition underlying the possibility of this most primitive form of perceptual judgment: namely, that things should be presented apart or in separation in order that any characterising judgment may be directed now to one and then again to another. Thus separation of presentment is a presupposition of cognition or judgment. Here I use the word 'presentment' not as equivalent to cognition, but as something presupposed in all—even the most primitive—acts of cognition. The

[1] As an illustration of how words lose their philological origin and become merely metaphorical, consider the expressions: 'There is a God,' 'There is an integer between 5 and 7.'

word 'pre*sent*'—with the accent on the second syllable[1]—is in English equivalent to 'give'; in this sense a *presentation* is eqivalent to *a datum*—where by 'datum' is meant not a piece of given knowledge, but a piece of given reality that is to be characterised in knowledge. Thus the presentation or the datum is what I have otherwise called the determinandum—that which is given or presented to thought to be thought about. This expresses briefly, the meaning of the primitive 'this.' The 'this' as thus defined is not rich in predicates and adjectives, but at the same time it cannot be said to be empty of adjectives or predicates, because, in the meaning of thisness, abstraction is made from all predicates or adjectives. But the 'this' cannot be explicated apart from an implicit reference to the 'that,' in the sense that the 'this' must be for the percipient presented in separation from the 'that': one determinandum is *one* to which *its own* adjectives may be assigned, just because *the other* must be presented in separation or apart from *the one*, before the most primitive form of articulate judgment is possible[2]. Briefly separateness is *before* relating; more specifically, it is the presupposition which makes it possible in more highly developed perception to define the relations (temporal or spatial) between those things which are first presented merely as separate.

[1] When accented on the *first* syllable, its meaning combines a reference both to space and to time; so that the word *presentation* contains in its *meaning* the three factors in our analysis, viz. the given, the here, and the now.

[2] It is here presumed that such mental processes as sense differentiation, etc., in which the experient is merely passive or recipient, must have been developed prior to the exercise of judgment, to furnish the material upon which the activity of thought can operate.

It is in this quite ultimate sense that I demur to Mr Bradley's dictum: 'distinction implies difference.' His dictum means, as far as I understand, what in my own terminology I should express by the phrase: 'otherness presupposes comparison' (the comparison, in particular, in which the relation of difference is asserted). Now in my view this dictum is exactly wrong: the assertion of 'otherness' does not presuppose or require a previous assertion of any relation of agreement or of difference. It does not even presuppose the possibility of asserting in the future any *particular* relation of agreement or of difference. The first important relation which will be elicited from otherness is, in fact, not any relation of agreement or difference at all, but a temporal or spatial relation; and thus the primitive assertion of otherness is only occasioned and rendered possible from the fact of separateness in presentation. When presentations are separate, then we can count one, two, three; further, we can connect them by temporal relations such as before and after, or by spatial relations such as above or below; and finally by relations of comparison such as like or unlike. These examples indicate my view of the quite primary nature of separateness of presentment, since it is for me the pre-requisite for *all* those acts of connecting with which logic or philosophy—and we may add psychology—is throughout concerned. In illustration, I have briefly referred only to relations of number, relations of time and space, and lastly to relations of comparison in a quite general sense.

Summarising this attempt to indicate the precise logical character of such primitive judgments as the exclamatory or impersonal, and their relation to more

highly developed judgments: we have found that to assert 'Lightning!' is to characterise, not reality as a whole, but a separate part of reality—to use an inadequate expression—and that the possibility for this primitive assertion to develop into higher, more inter-related forms of judgment, is wholly dependent on the attribution of an adjective to a part of reality presented in separation from other presentables.

For the purpose of further elucidation we may bring two or three assertions into connection with one another, which might be briefly formulated thus: 'Lightning now!' 'Lightning again!' 'Thunder then!' The first two judgments when connected, involve two manifestations of the same character denominated lightning, which are *two* because they have been *separately* presented. The use of the terms 'now' and 'then' does not necessarily presuppose a developed system of temporal relations; but they indicate at least the possibility of defining relations in time between separately presented manifestations. Again the exclamation 'Thunder!' when taken in connection with the exclamation 'Lightning!' already presupposes—not only that the manifestations are given somehow in separation—but further that the percipient has characterised the separated manifestations by different adjectives. I will not here discuss whether these *manifestations* (as I have called them) are, in their primitive recognition, merely the individual's sense-experiences of sound and light, or whether from first to last they are something other than sense-experiences. In either case our logical point will be the same, when it is agreed that they are given separately, and that their separate presentment is the

precondition for any further development of thought or of perception. The view put forward here is so far equivalent to Kant's in that I regard space and time as the conditions of the otherness of sense-experiences upon which the possibility of cognising *determinate* spatial and temporal relations depends, and that this characteristic of space and time is what constitutes Sense-Experience into a manifold, i.e. a plurality of experiences, which we can proceed to count as many only because of their separate presentment.

Taking more elaborate examples of these primitive forms of perceptual judgment: 'This is a flash of lightning,' 'This (same) flash of lightning is brighter than that (other),' 'This (same) flash appeared before that clap of thunder'; we note that in the predesignations 'this' and 'that' the percipient has passed beyond the indefinite article 'a,' and has identified a certain manifestation as that of which more than one characterisation can be predicated—e.g. 'lightning' and 'brighter than that.' It is this identification which gives to the articles 'this' and 'that' a significance which may be called referential, to be distinguished from their use as demonstratives; and in this alternation between the demonstrative and the referential usage, we can trace, I think, the very primitive way in which thought develops: first, in fixing attention upon a phenomenon by pointing to its position; and, next, in identifying it as the same in character when it is changing its spatial relations. All that is theoretically required for identification is the retention—or rather the detention—of our cognition or judgment upon a certain manifestation; but, when attributing *different* qualities or relations to what

continues to function as the *same* logical subject, we are assisted by the temporal continuance of a phenomenon, either with unaltered quality, or in an unaltered position, or in a *continuously* changing position, etc. Thus the changes involving variations in space, time, and quality amongst different manifestations of the same phenomenon constitute the groundwork upon which the several judgments of relation are built.

In asserting 'There was a flash of lightning that was very brilliant and that preceded a clap of thunder' we are grasping the identity of a certain manifestation, thus used in two propositions, of which one predicates a relation in time to a clap of thunder, and the other, a quality characterising the flash itself. Any such connected judgment contains implicitly the relation of identity, in that the manifestation is maintained as an object in thought, while we form two judgments with respect to it. It is only, in short, in the act of joining two different characterisations that any meaning for identity can be found. In an elementary judgment which predicates only *one* adjective, no scope or significance for the notion of identifying a subject as such can be afforded. Thus the three factors in the thinking process which the 'this' reveals are: (1) the given—which is equivalent to the 'it' in 'It lightens!' (2) the demonstrative—which, by indicating spatial position, helps towards unique identification, (3) the referential—which marks the achievement of this process of identification. As will be seen from the discussion in a subsequent chapter, these three elements of significance in the 'this' bring it into line with the proper name.

CHAPTER III

COMPOUND PROPOSITIONS

§ 1. Having examined the proposition in its more philosophical aspects, and in particular from the point of view of its analysis, the present chapter will be mainly devoted to a strictly formal account of the proposition, and will be entirely concerned with the synthesis of propositions considered apart from their analysis. The chapter is intended to supply a general introduction to the fundamental principles of Formal Logic; and formulae will first be laid down without any attempt at criticism or justification—which will be reserved for subsequent discussion. For this purpose we begin by considering the different ways in which a new proposition may be constructed out of one or more given propositions.

In the first place, given a *single* proposition, we may construct its negative—expressed by the prefix *not*—not-p being taken as equivalent to p-false. Next, we consider the construction of a proposition out of two or more given propositions. The proposition thus constructed will be called *compound*, and the component propositions out of which it is constructed, may be called *simple*, relatively to the compound, although they need not be in any absolute sense simple. The prefix *not* may be attached, not only to any simple proposition, but also to a compound proposition, any of whose components again may be negative.

The different forms that may be assumed by compound propositions are indicated by different *conjunctions*. A proposition in whose construction the only formal elements involved are negation and the logical conjunctions is called a *Conjunctional Function* of its component propositions. The term conjunctional function must be understood to include functions in which negation *or* any one or more of the logical conjunctions is absent. We have to point out that the compound proposition is to be regarded, not as a mere plurality of propositions, but as a *single* proposition, of which truth or falsity can be significantly predicated irrespectively of the truth or falsity of any of its several components. Furthermore, the *meaning* of each of the component propositions must be understood to be assignable irrespectively of the compound into which it enters, so that the meaning which it is understood to convey when considered in isolation is unaffected by the mode in which it is combined with other propositions.

§ 2. We will proceed to enumerate the several modes of logical conjunction by which a compound proposition may be constructed out of two component propositions, say p and q. Of all such modes of conjunction, the most fundamental is that expressed by the word *and*: this mode will be called par excellence *conjunctive*, and the components thus joined will be called *conjuncts*. Thus the compound propositions—

(*a*) 'p and q,' (*b*) 'p and not-q,' (*c*) 'not-p and q,' (*d*) 'not-p and not-q,'

are the conjunctive functions of the conjuncts p, q; p, not-q; not-p, q; not-p, not-q; respectively. There are thus four distinct conjunctive forms of proposition involving the two propositions p, q, taken positively or negatively.

The significance of the conjunctional *and* will be best understood in the first instance, by contrasting it with the enumerative *and*. For example, we use the merely enumerative *and* when we speak of constructing any compound proposition out of the components *p and q*. Here we are not specifying any mode in which *p* and *q* are to be combined so as to constitute one form of unity rather than another; we are treating the components (so to speak) severally, not combinatorially. In other words, the enumeration—*p* and *q*—yields two propositions, the enumeration—*p* and *q* and *r*—yields three propositions, etc.; but the conjunctive '*p* and *q*,' or the conjunctive '*p* and *q* and *r*' etc., yields one proposition. Again, of the enumerated propositions—*p* and *q* and *r* and...—some may be true and others false; but the conjunctive proposition '*p* and *q* and *r* and...' must be either definitively true or definitively false. Thus in *conjoining* two or more propositions we are realising, not merely the force of each considered separately, but their joint force. The difference is conclusively proved from the consideration that we may infer from the conjunctive proposition '*p* and *q*' a set of propositions none of which could be inferred from *p* alone or *q* alone. The same holds, of course, where three or more conjuncts are involved: thus, with *p*, *q*, *r*, as components, seven distinct groups[1] of propositions are generated: viz. the three groups consisting of propositions implied by *p*, by *q*, by *r* respectively; the three groups consisting of propositions implied by '*p* and *q*,' by '*p* and *r*,' by '*q* and *r*' respectively; and lastly, the group consisting of propositions implied by '*p* and *q* and *r*.'

[1] The term *group* is here used in its precise mathematical significance.

§ 3. In our first presentation of formal principles we shall introduce certain familiarly understood notions, such as equivalence, inference etc., without any attempt at showing how some of them might be defined in terms of others. The same plan will be adopted in regard to the question of the demonstrability of the formal principles themselves; these will be put forward as familiarly acceptable, without any attempt at showing how some of them might be proved by means of others. Ultimately, certain *notions* must be taken as intelligible without definition, and certain *propositions* must be taken as assertible without demonstration. All other notions (intrinsically logical) will have to be *defined* as dependent upon those that have been put forward without definition; and all other propositions (intrinsically logical) will have to be *demonstrated* as dependent upon those that have been put forward without demonstration. But we shall not, in our first outline, raise the question of the dependence or independence of the notions and propositions laid down.

Thus the formal law which holds of Negation is called the Law of Double Negation: viz. not-not-$p \equiv p$.

§ 4. We now lay down the formal laws which hold of compound propositions constructed by means of the conjunction *and*. They are as follows:

Laws of Conjunctive Propositions

1. The Reiterative Law:

$$p \text{ and } p \equiv p.$$

2. The Commutative Law:

$$p \text{ and } q \quad q \text{ and } p.$$

3. The Associative Law:

$$(p \text{ and } q) \text{ and } r \equiv p \text{ and } (q \text{ and } r).$$

Here the notion of *equivalence* (expressed by the short-hand symbol ≡) is taken as ultimate and therefore as not requiring to be defined. These laws and similar formal principles are apt to be condemned as trivial. Their significance will be best appreciated by reverting to the distinction between the mental acts of assertion and progression in thought on the one hand, and the propositions to which thought is directed on the other. Thus the laws above formulated indicate, in general, equivalence as regards the propositions asserted, in spite of variations in the modes in which they come before thought. Thus the content of what is asserted is not affected, firstly, by any *re*-assertion; nor, secondly, by any different *order* amongst assertions; nor, thirdly, by any different *grouping* of the assertions.

§ 5. Having considered the *Conjunctive* form of proposition, we turn next to the consideration of the remaining fundamental conjunctional forms. These will be classed under the one head *Composite* for reasons which will be apparent later. So far, compound propositions have been divided into the two species *Conjunctive* and *Composite*, and we shall now proceed to subdivide the latter into four sub-species, each of which has its appropriate conjunctional expression, viz.:

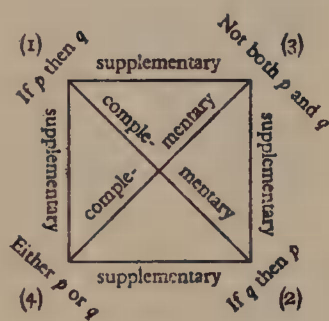
(1) The Direct-Implicative function of p, q:— If p then q.
(2) The Counter-Implicative function of p, q:— If q then p.
(3) The Disjunctive function of p, q:— Not-both p and q.
(4) The Alternative function of p, q:— Either p or q.

In the implicative function 'If p then q,' p is implicans[1] and q implicate; in the counter-implicative function 'If q then p,' p is implicate and q is implicans[1]; in the dis-

[1] The plural of *implicans* must be written: *implicants*.

junctive function 'Not-both p and q,' p and q are disjuncts; and in the alternative function 'Either p or q,' p and q are alternants. These four functions of p, q, are distinct and independent of one another. The technical names that have been chosen are obviously in accordance with ordinary linguistic usage. The implicative and counter-implicative functions are said to be *Complementary* to one another, as also the disjunctive and alternative functions. Each of the four other pairs, viz. (1) and (3); (1) and (4); (2) and (3); (2) and (4) may be called a pair of *Supplementary* propositions. These names are conveniently retained in the memory by help of

The Square of Independence



Now when we bring into antithesis the four conjunctive functions:

(1) p and not-q; (2) not-p and q; (3) p and q; (4) not-p and not-q;

with the four composite functions:

(1) if p then q; (2) if q then p; (3) not-both p and q; (4) either p or q;

we shall find that each of the composite propositions is equivalent to the negation of the corresponding conjunctive. This is directly seen in the case (3) of the conjunctive and the disjunctive functions of p, q. Thus,

'Not-both p and q' is the direct negative of 'Both p and q.' Again, for case (4) 'Either p or q' is the obvious negative of 'Neither p nor q.' The relations of negation for *all* cases may be derived by first systematically tabulating the equivalences which hold amongst the composite functions, as below: abbreviating not-p and not-q into the forms $\bar{p}$ and $\bar{q}$ respectively.

Table of Equivalences of the Composite Functions

	Implicative Form		Counter-Implicative Form		Disjunctive Form		Alternative Form
1.	If p then q	=	If $\bar{q}$ then $\bar{p}$	=	Not both p and $\bar{q}$	=	Either $\bar{p}$ or q
2.	If $\bar{p}$ then $\bar{q}$	=	If q then p	=	Not both $\bar{p}$ and q	=	Either p or $\bar{q}$
3.	If p then $\bar{q}$	=	If q then $\bar{p}$	=	Not both p and q	=	Either $\bar{p}$ or $\bar{q}$
4.	If $\bar{p}$ then q	=	If $\bar{q}$ then p	=	Not both $\bar{p}$ and $\bar{q}$	=	Either p or q

In the above table it will be observed:

(*a*) That each composite function can be expressed in four equivalent forms: thus, any two propositions in the same row are equivalent, while any two propositions in different rows are distinct and independent.

(*b*) That the propositions represented along the principal diagonal are expressed in terms of the *positive* components p, q; being in fact identical respectively with the implicative, the counter-implicative, the disjunctive and the alternative functions of p, q.

(*c*) That all the remaining propositions are expressed as functions of p and not-q, or of not-p and q, or of not-p and not-q.

We may translate the equivalences tabulated above in the form of equivalences of functions, thus:

1. The implicative function of p, q; the counter-implicative function of not-p, not-q; the disjunctive func-

tion of p, not-q; and the alternative function of not-p, q, are all equivalent to one another. Again,

2. The counter-implicative function of p, q; the implicative function of not-p, not-q; the disjunctive function of not-p, q; and the alternative function of p, not-q, are all equivalent to one another. Again,

3. The disjunctive function of p, q; the alternative function of not-p, not-q; the implicative function of p, not-q; and the counter-implicative function of not-p, q, are all equivalent to one another. Again,

4. The alternative function of p, q; the disjunctive function of not-p, not-q; the implicative function of not-p, q; and the counter-implicative function of p, not-q, are all equivalent to one another.

Since the force of each of the four composite functions of p, q can be represented by using either the Implicative or the Counter-implicative or the Disjunctive or the Alternative form, the classification of the four functions under one head *Composite* is justified. And since each Composite function is equivalent to a certain *Disjunctive* proposition, it is also equivalent to the negation of the corresponding *Conjunctive* proposition. Thus:

1. The implicative	'If p then q'	negates	the	Conjunctive	'p and $\bar{q}$.'
2. The counter-implicative	'If q then p'	„	„	„	'$\bar{p}$ and q.'
3. The disjunctive	'Not both p and q'	„	„	„	'p and q.'
4. The alternative	'Either p or q'	„	„	„	'$\bar{p}$ and $\bar{q}$.'

Thus inasmuch as *no* composite function is *equivalent* to any conjunctive function, we have justified our division of compound propositions into the two fundamentally opposed species Conjunctive and Composite.

The distinction and relation between these composite forms of proposition may be further brought out by tabulating the inferences in which a simple conclusion is drawn from the conjunction of a composite with a simple premiss. In traditional logic, the Latin verbs *ponere* (to lay down or *assert*) and *tollere* (to raise up or *deny*) have been used in describing these different modes of argument. The gerund *ponendo* (by affirming) or *tollendo* (by denying) indicates the nature of the (simple) *premiss* that occurs; while the participle *ponens* (affirming) or *tollens* (denying) indicates the nature of the (simple) *conclusion* that occurs: the validity or invalidity of the argument depending on the nature of the *composite* premiss. There are therefore four modes to be considered corresponding to the four varieties of the composite proposition, thus:

Table of Valid Modes

Modus		Form of Composite Premiss
1. Ponendo Ponens:	If p then q; but p; $\therefore q$	The Implicative.
2. Tollendo Tollens:	If q then p; but $\bar{p}$; $\therefore \bar{q}$	The Counter-Implic.
3. Ponendo Tollens:	Not both p and q; but p; $\therefore \bar{q}$	The Disjunctive.
4. Tollendo Ponens:	Either p or q; but $\bar{p}$; $\therefore q$	The Alternative.

The customary fallacies in inferences of this type may be exhibited as due to the confusion between a composite proposition and its complementary:

Table of Invalid Modes

Modus		Form of Composite Premiss
1. Ponendo Ponens:	If q then p; but p; $\therefore q$	The Counter-Implic.
2. Tollendo Tollens:	If p then q; but $\bar{p}$; $\therefore \bar{q}$	The Implicative.
3. Ponendo Tollens:	Either p or q; but p; $\therefore \bar{q}$	The Alternative.
4. Tollendo Ponens:	Not both p and q; but $\bar{p}$; $\therefore q$	The Disjunctive.

The rules for correct inference from the above table of valid modes may be thus stated:

1. From an implicative, combined with the affirmation of its implicans, we may infer the affirmation of its implicate.

2. From an implicative, combined with the denial of its implicate, we may infer the denial of its implicans.

3. From a disjunctive, combined with the affirmation of one of its disjuncts, we may infer the denial of the other disjunct.

4. From an alternative, combined with the denial of one of its alternants, we may infer the affirmation of the other alternant.

§ 6. We ought here to refer to an historic controversy as regards the interpretation of the conjunction 'or.' It has been held by one party of logicians that what I have called the Alternative form of proposition, viz., that expressed by either-or, should be interpreted so as to include what I have called the Disjunctive, viz., that expressed by not-both. This view has undoubtedly been (perhaps unwittingly) fostered by the almost universal misemployment of the term *Disjunctive* to stand for what ought to be called Alternative. This prevalent confusion in terminology has led to a real blunder committed by logicians. The blunder consists in the fallacious use of the Ponendo Tollens as exhibited in the table above given. Consider the argument: '*A* will be either first or second'; 'It is found that *A* is second'; therefore '*A* is not first.' Here the conclusion is represented as following from the promising qualifications of the candidate *A*, whereas it really follows from the premiss '*A* cannot be both first and second.' In fact, the Alternative

proposition which is put as premiss is absolutely irrelevant to the conclusion, which would be equally correctly inferred whether the alternative predication were false or true.

It remains then to consider whether the logician can properly impose the one interpretation of the alternative form of proposition rather than the other. The reply here, as in other similar cases, is that, in the matter of verbal interpretation, the logician can impose legislation—not upon others—but only upon himself. However, where any form of verbal expression is admittedly ambiguous, it is better to adopt the interpretation which gives the smaller rather than the greater force to a form of proposition, since otherwise there is danger of attaching to the judgment an item of significance beyond that intended by the asserter. This principle of interpretation has the further advantage that it compels the speaker when necessary to state unmistakeably and explicitly what may have been implicitly and perhaps confusedly present in his mind. I have therefore adopted as my interpretation of the form *Either-or* that smaller import according to which it does *not* include *Not-both*. Those logicians who have insisted on what is called the 'exclusive' interpretation of the alternative form of proposition (i.e. the interpretation according to which *Either-or* includes *Not-both*) seem sometimes to have been guilty of a confusion between what a proposition asserts, and what may happen to be known independently of the proposition. Thus it may very well be the case that the alternants in an alternative proposition are almost always 'exclusive' to one another; but this, so far from proving that the alternative proposition *affirms*

this exclusiveness, rather suggests that the exclusiveness is a fact commonly known independently of the special information supplied by the alternative proposition itself.

In this connection, the significance of the term complementary which I have applied to the implicative and counter-implicative as well as to the disjunctive and alternative, may be brought out. Propositions are appropriately called complementary when a special importance attaches to their conjoint assertion[1]. Thus it may be regarded as an ideal of science to establish a pair of propositions in which the implicans of the one is the implicate of the other; and again to establish a number of propositions which are mutually co-disjunct and collectively co-alternate. The term complementary is especially applicable where propositions are conjoined in either of these ways, because separately the propositions represent the fact partially, and taken together they represent the same fact with relative completeness.

We next consider the inferences that can be drawn from the conjunction of two *supplementary* propositions. These may be tabulated in two forms, the first of which brings out the fundamental notion of the *Dilemma*; and the second that of the *Reductio ad Impossibile*.

First Table for the Conjunction of Supplementaries
The Dilemma

(1) 'If p then q' and (4) 'If $\bar{p}$ then q': therefore, q.
(3) 'If p then $\bar{q}$' and (2) 'If $\bar{p}$ then $\bar{q}$': therefore, $\bar{q}$.
(2) 'If q then p' and (4) 'If $\bar{q}$ then p': therefore, p.
(3) 'If q then $\bar{p}$' and (1) 'If $\bar{q}$ then $\bar{p}$': therefore, $\bar{p}$.

[1] Thus complementary propositions might be defined as those which are frequently confused in thought and frequently conjoined in fact.

The above table illustrates the following principle:

The conjunction of two implicatives, containing a common implicate but contradictory implicants, yields the affirmation of the simple proposition standing as common implicate. Or otherwise:

Any proposed proposition must be true when its truth would be implied both by the supposition of the truth and by the supposition of the falsity of some other proposition.

Second Table for the Conjunction of Supplementaries
The Reductio ad Impossibile

(1) 'If $\bar{q}$ then $\bar{p}$' and (4) 'If $\bar{q}$ then p': therefore, q.
(3) 'If q then $\bar{p}$' and (2) 'If q then p': therefore, $\bar{q}$.
(2) 'If $\bar{p}$ then $\bar{q}$' and (4) 'If $\bar{p}$ then q': therefore, p.
(3) 'If p then $\bar{q}$' and (1) 'If p then q': therefore, $\bar{p}$.

This second table illustrates the following principle:

The conjunction of two implicatives, containing a common implicans but contradictory implicates, yields the denial of the simple proposition standing as common implicans. Or otherwise:

Any proposed proposition must be false when the supposition of its truth would imply (by one line of argument) the truth and (by another line of argument) the falsity of some other proposition.

§ 7. In tabulating the formulae for Composite propositions as above I have merely systematised (with slight extensions and modifications of terminology) what has been long taught in traditional logic; and it is only in these later days that criticisms have been directed against the traditional formulae, especially on the ground that their uncritical acceptance has been found

to lead to certain paradoxical consequences, which may be called the Paradoxes of Implication. In this connection it is (I think) desirable to explain what is meant by a paradox. When a thinker accepts step by step the principles or formulae propounded by the logician until a formula is reached which conflicts with his common-sense, then it is that he is confronted with a paradox. The paradox arises—not from a merely blind submission to the authority of logic, or from any arbitrary or unusual use of terms on the logician's part—but from the very nature of the case, as apprehended in the exercise of powers of reasoning with which everyone is endowed. In particular, the paradoxes of implication are not due to any unnatural use of the *term* implication, nor to the positing of any *fundamental* formula that appears otherwise than acceptable to common sense. It is the formulae that are *derived*—by apparently unexceptionable means from apparently unexceptionable first principles—that appear to be exceptionable.

Let us trace the steps by which we reach a typical paradox. Consider the alternative 'Not-p or q.' If this alternative were conjoined with the assertion 'p,' we should infer 'q.' Hence, 'Not-p or q' is equivalent to 'If p then q.' Similarly 'p or q' is equivalent to 'If not-p then q.' Now it is obvious that the less determinate statement 'p or q' could always be inferred from the more determinate statement 'p': e.g. from the relatively determinate statement 'A is a solicitor' we could infer 'A is a solicitor or a barrister' i.e. 'A is a lawyer.' Hence, whatever proposition q may stand for, we can infer 'p or q' from 'p'; or again, whatever p may stand

for we can infer 'not-p or q' from q. Hence (i) given 'p' we may infer 'If not-p then q,' and (ii) given 'q' we may infer 'If p then q,' whatever propositions p and q may stand for. These two consequences of the uncritical acceptance of traditional formulae have been expressed thus: (i) A *false* proposition (e.g. not-p when p has been asserted) implies *any* proposition (e.g. q); (ii) A *true* proposition (e.g. q, when q has been asserted) is implied by *any* proposition (e.g. p). Thus '$2+3=7$' would imply that 'It will rain to-morrow'; and 'It will rain to-morrow' would imply that '$2+3=5$.' That these two implicative statements are technically correct is shown by translating them into their equivalent alternative forms, viz.: (i) 'Either $2+3$ is unequal to 7 or it will rain to-morrow'; (ii) 'Either it will not rain to-morrow or $2+3=5$.' We may certainly say that one or other of the two alternants in (i) as also in (ii) is true, the other being of course doubtful.

Taking 'If p then q' to stand for the paradoxically reached implicative in both cases, we have shown that (i) from the denial of p (the implicans), and (ii) from the affirmation of q (the implicate) we may pass to the assertion 'If p then q.' This is, of course, only another way of saying that the implicative 'If p then q' is equivalent to the alternative 'p false or q true.' Thus when we know that 'If p then q' is true, it follows that we know that 'either p is false or q is true'; but it does *not* follow that either 'we know that p is false' or 'we know that q is true.' The paradoxically reached implicative merely brings out the fact that this may be so *in some cases*: i.e. when asserting 'If p then q,' there are cases in which we know that 'p is false,' and there

are cases in which we know that 'q is true.' But it is proper to enquire whether in actual language—literary or colloquial—the implicative form of proposition is ever introduced in this paradoxical manner. On the one hand, we find such expressions as: 'If that boy comes back, I'll eat my head'; 'If you jump over that hedge, I'll give you a thousand pounds'; 'If universal peace is to come tomorrow, the nature of mankind must be very different from what philosophers, scientists and historians have taken it to be'; etc., etc. Such phrases are always interpreted as expressing the *speaker's* intention to *deny the implicans*; the reason being that the *hearer* is assumed to be ready to deny the *implicate*. Again, on the other hand, we find such forms as: 'If Shakespeare knew no Greek, he was not incapable of creating great tragedies.' 'If Britain is a tiny island, on the British Empire the sun never sets.' 'If Boswell was a fool, he wrote a work that will live longer than that of many a wiser man.' 'If Lloyd George has had none of the advantages of a public school education, it cannot be maintained that he is an unintelligent politician.' Such phrases are always interpreted as expressing the *speaker's* intention to *affirm the implicate*, the reason being that the *hearer* may be assumed to be willing to *affirm the implicans*.

Looking more closely into the matter we find that when a speaker adopts the implicative form to express his denial of the implicans, he tacitly expects his hearer to supplement his statement with a *tollendo tollens*; and when he adopts it to express his affirmation of the implicate, he expects the hearer to supplement it with a *ponendo ponens*. Furthermore, inasmuch as the alternative form of proposition requires to be supplemented

by a *tollendo* (*ponens*) and the disjunctive by a *ponendo* (*tollens*), we find that an implicative intended to express the denial of its implicans is quite naturally expressed otherwise as an alternative: e.g. 'That boy won't come back or I'll eat my head,' to which the hearer is supposed to add 'But you won't eat your head'; therefore (I am to believe that) 'the boy won't come back' (*tollendo ponens*); and we find that an implicative intended to express the affirmation of its implicate is quite naturally expressed otherwise as a disjunctive: e.g. 'It cannot be held that Shakespeare both knew no Greek and was incapable of creating great tragedies,' to which the hearer is supposed to add 'But Shakespeare knew no Greek,' and therefore (I am to believe that) 'he was capable of creating great tragedies' (*ponendo tollens*).

We have yet to explain how the appearance of paradox is to be removed in the general case of a composite being inferred from the denial of an implicans (or disjunct) or from the affirmation of an implicate (or alternant). Now the ordinary purpose to which an implicative (or, more generally, a composite) proposition is put is *inference*: so much so that most persons would hesitate to assert the relation expressed in a composite proposition unless they were prepared to use it for purposes of inference in one or other of the four modes, *ponendo ponens*, etc. In other words, Implication is naturally regarded as tantamount to Potential Inference. Now when (i) we have inferred 'If p then q' from the denial of 'p,' can we proceed from 'If p then q' conjoined with 'p' to infer 'q'? In this case we join the affirmation of 'p' with a premiss which has been inferred from the denial of p; and this involves *Contra-*

diction, so that such an inference is impossible. Again, when (ii) we have inferred 'If p then q' from the affirmation of 'q,' can we proceed from 'If p then q' conjoined with 'p' to infer 'q'? In this case we profess to infer 'q' by means of a premiss which was itself inferred from 'q'; and this involves *Circularity*, so that this inference again must be rejected. The solution of the paradox is therefore found in the consideration that though we may correctly infer an implicative from the denial of its implicans, or from the affirmation of its implicate, or a disjunctive from the denial of one of its disjuncts, or an alternative from the affirmation of one of its alternants, yet the implicative, disjunctive or alternative so reached cannot be applied for purposes of further inference without committing the logical fallacy either of contradiction or of circularity. Now it must be observed that the rhetorical or colloquial introduction of a paradoxical composite, which is meant to be interpreted as the simple affirmation or denial of one of its components, achieves its intention by introducing—as the other component of the composite—a proposition whose falsity or truth (as the case may be) is palpably obvious to the hearer. The hearer is then expected to supplement the composite by joining it with the obvious affirmation or denial of the added component, and thereby, in interpreting the intention of the speaker, to arrive at the proposition as conclusion which the speaker took as his first premiss. Accordingly the process of interpretation consists in taking the same propositions in the same mode and arrangement as would have entailed circularity if adopted by the speaker.

§ 8. The distinction between an implicative pro-

position that can and one that cannot be used for inferential purposes may now be further elucidated by reference to the distinction between Hypothesis and Assertion. In order that an implicative may be used for inference, both the implicans and the implicate must be entertained hypothetically. In the case of *ponendo ponens* the process of inference consists in passing to the *assertion* of the implicate by means of the *assertion* of the implicans, so that the propositions that were entertained hypothetically in the implicative, come to be adopted assertively in the process of inference. The same holds, *mutatis mutandis*, for the other modes. Now when we have inferred an implicative from the affirmation of its implicate or from the denial of its implicans—as in the case of the implicative which appears paradoxical—the two components of the implicative thus reached cannot both be regarded as having been entertained hypothetically; and hence the principle according to which inference is a process of passing from propositions entertained hypothetically to the same propositions taken assertorically, would be violated if we used the composite for inference. This consideration constitutes a further explanation of how the paradoxes in question are solved.

The above analysis may be symbolically represented by placing under the letter standing for a proposition the sign $\vdash$ to stand for assertorically adopted and the sign H for hypothetically entertained.

Thus the fundamental formula for correct inference may be rendered:

From '$\underset{\text{H}}{p}$ would imply $\underset{\text{H}}{q}$' with $\underset{\vdash}{p}$; we may infer $\underset{\vdash}{q}$,

where, in the implicative premiss, both implicans and implicate are entertained hypothetically.

Now the following inferences, which lead to paradoxical consequences, may be considered correct: i.e.

(*a*) From $\underset{\vdash}{q}$, we may infer '$\underset{\mathrm{H}}{p}$ would imply $\underset{\vdash}{q}$.'

(*b*) From $\underset{\vdash}{\bar{p}}$, we may infer '$\underset{\vdash}{p}$ would imply $\underset{\mathrm{H}}{q}$.'

But the implicative conclusions here reached cannot be used for further inference: i.e.

(*c*) From '$\underset{\mathrm{H}}{p}$ would imply $\underset{\vdash}{q}$' with $\underset{\vdash}{p}$; we cannot infer $\underset{\vdash}{q}$.

(*d*) From '$\underset{\vdash}{p}$ would imply $\underset{\mathrm{H}}{q}$' with $\underset{\vdash}{p}$; we cannot infer $\underset{\vdash}{q}$.

For in (*c*) the implicate, and in (*d*) the implicans enters assertorically, and these inferences therefore contravene the above fundamental formula which requires that both implicate and implicans should enter hypothetically. Thus while admitting (*a*) that 'a true proposition would be implied by *any* proposition,' yet we cannot admit (*c*) that 'a true proposition *can be inferred* from any proposition.' Similarly, while admitting (*b*) that 'a false proposition would imply *any* proposition,' yet we cannot admit (*d*) that 'from a false proposition we can *infer* any proposition.' In fact, the attempted inference (*c*), where the conclusion has already been asserted, would entail *circularity*; and the attempted inference (*d*), where the premiss has already been denied, would involve *contradiction*.

Still maintaining the equivalence of the composite propositions expressible in the implicative, the counter-implicative, the alternative or the disjunctive form, each of these four forms will give rise to a like paradox. The

following table gives all the cases in which we reach a Paradoxical Composite; that is, a Composite which cannot be used for inference, either in the modus *ponendo ponens*, *tollendo tollens*, *ponendo tollens* or *tollendo ponens*. The sign of assertion in each composite must be interpreted to mean asserted to be true when the term to which it is attached *agrees* with the premiss, and asserted to be false when it *contradicts* the premiss.

Table of Paradoxical Composites

(*a*) From $\underset{\vdash}{q}$ we may properly infer

(1) $\underset{\mathrm{H}}{\bar{p}}$ or $\underset{\vdash}{q}$ = If $\underset{\mathrm{H}}{p}$ then $\underset{\vdash}{q}$ = If $\underset{\vdash}{\bar{q}}$ then $\underset{\mathrm{H}}{\bar{p}}$ = Not both $\underset{\mathrm{H}}{p}$ and $\underset{\vdash}{\bar{q}}$,

or (2) $\underset{\mathrm{H}}{p}$ or $\underset{\vdash}{q}$ = If $\underset{\mathrm{H}}{\bar{p}}$ then $\underset{\vdash}{q}$ = If $\underset{\vdash}{\bar{q}}$ then $\underset{\mathrm{H}}{p}$ = Not both $\underset{\mathrm{H}}{\bar{p}}$ and $\underset{\vdash}{\bar{q}}$.

(*b*) From $\underset{\vdash}{\bar{q}}$ we may properly infer

(3) $\underset{\mathrm{H}}{p}$ or $\underset{\vdash}{\bar{q}}$ = If $\underset{\mathrm{H}}{\bar{p}}$ then $\underset{\vdash}{\bar{q}}$ = If $\underset{\vdash}{q}$ then $\underset{\mathrm{H}}{p}$ = Not both $\underset{\mathrm{H}}{\bar{p}}$ and $\underset{\vdash}{q}$,

or (4) $\underset{\mathrm{H}}{\bar{p}}$ or $\underset{\vdash}{\bar{q}}$ = If $\underset{\mathrm{H}}{p}$ then $\underset{\vdash}{\bar{q}}$ = If $\underset{\vdash}{q}$ then $\underset{\mathrm{H}}{\bar{p}}$ = Not both $\underset{\mathrm{H}}{p}$ and $\underset{\vdash}{\bar{q}}$.

The above composites can never be used for further inference. Thus:

in line (1), the attempted inference
'$p \therefore q$' would be circular and 'not-$q \therefore$ not-p' would be contradictory;
in line (2), the attempted inference
'not-$p \therefore q$' would be circular, and 'not-$q \therefore p$' would be contradictory;
in line (3), the attempted inference
'not-$p \therefore$ not-q' would be circular, and '$q \therefore p$' would be contradictory;
in line (4), the attempted inference
'$p \therefore$ not-q' would be circular, and '$q \therefore$ not-p' would be contradictory.

The paradox of implication assumes many forms, some of which are not easily recognised as involving mere varieties of the same fundamental principle. But

I believe that they can all be resolved by the consideration that we cannot *without qualification* apply a composite and (in particular) an implicative proposition to the further process of inference. Such application is possible only when the composite has been reached irrespectively of any *assertion* of the truth or falsity of its components. In other words, it is a necessary condition for further inference that the components of a composite should really have been entertained hypothetically when asserting that composite.

§ 9. The theory of compound propositions leads to a special development when in the conjunctives the components are taken—not, as hitherto, assertorically—but hypothetically as in the composites. The conjunctives will now be naturally expressed by such words as possible or compatible, while the composite forms which respectively contradict the conjunctives will be expressed by such words as necessary or impossible. If we select the negative form for these conjunctives, we should write as contradictory pairs:

Conjunctives (*possible*)	*Composites* (*necessary*)
$\bar{a}$. p does not imply q	a. p implies q
$\bar{b}$. p is not implied by q	b. p is implied by q
$\bar{c}$. p is not co-disjunct to q	c. p is co-disjunct to q
$\bar{d}$. p is not co-alternate to q	d. p is co-alternate to q

Or otherwise, using the term 'possible' throughout, the four conjunctives will assume the form that the several conjunctions—$p\bar{q}$, $\bar{p}q$, pq and $\bar{p}\bar{q}$—are respectively *possible*. Here the word *possible* is equivalent to being merely hypothetically entertained, so that the several conjunctives are now qualified in the same way as are the simple components themselves. Similarly the four

corresponding composites may be expressed negatively by using the term 'impossible,' and will assume the form that the *con*junctions $p\bar{q}$, $\bar{p}q$, pq and $\bar{p}\bar{q}$ are respectively *impossible*, or (which means the same) that the *dis*junctions $p\bar{q}$, $\bar{p}q$, pq and $\bar{p}\bar{q}$ are *necessary*. Now just as 'possible' here means merely 'hypothetically entertained,' so 'impossible' and 'necessary' mean respectively 'assertorically denied' and 'assertorically affirmed.'

The above scheme leads to the consideration of the determinate relations that could subsist of p to q when these eight propositions (conjunctives and composites) are combined in every possible way without contradiction. *Primâ facie* there are 16 such combinations obtained by selecting a or $\bar{a}$, b or $\bar{b}$, c or $\bar{c}$, d or $\bar{d}$ for one of the four constituent terms. Out of these 16 combinations, however, some will involve a conjunction of supplementaries (see tables on pp. 37, 38), which would entail the assertorical affirmation or denial of one of the components p or q, and consequently would not exhibit a *relation* of p to q. The combinations that, on this ground, must be disallowed are the following *nine*:

$$a\bar{b}c\bar{d},\ a\bar{b}\bar{c}d,\ \bar{a}b\bar{c}d,\ \bar{a}bc\bar{d};\ \bar{a}bcd,\ \bar{b}acd,\ \bar{c}abd,\ \bar{d}abc;\ abcd.$$

The combinations that remain to be admitted are therefore the following *seven*:

$$ab\bar{c}\bar{d},\ cd\bar{a}\bar{b};\ a\bar{b}\bar{c}\bar{d},\ b\bar{a}\bar{c}\bar{d},\ c\bar{d}\bar{a}\bar{b},\ d\bar{c}\bar{a}\bar{b};\ \bar{a}\bar{b}\bar{c}\bar{d}.$$

In fact, under the imposed restriction, since a or b cannot be conjoined with c or d, it follows that we must always conjoin a with $\bar{c}$ and $\bar{d}$; b with $\bar{c}$ and $\bar{d}$; c with $\bar{a}$ and $\bar{b}$; d with $\bar{a}$ and $\bar{b}$. This being understood, the

seven permissible combinations that remain are properly to be expressed in the more simple forms:

$$ab,\ cd;\ a\bar{b},\ b\bar{a},\ c\bar{d},\ d\bar{c};\ \text{and}\ \bar{a}\bar{b}\bar{c}\bar{d}.$$

These will be represented (but re-arranged for purposes of symmetry) in the following table giving all the possible relations of any proposition p to any proposition q. The technical names which I propose to adopt for the several relations are printed in the second column of the table.

Table of possible relations of proposition p to proposition q.

1. (a, b): p implies and is implied by q.	p is co-implicant to q.
2. $(a, \bar{b})$: p implies but is not implied by q.	p is super-implicant to q.
3. $(b, \bar{a})$: p is implied by but does not imply q.	p is sub-implicant to q.
4. $(\bar{a}, \bar{b}, \bar{c}, \bar{d})$: p is neither implicans nor implicate nor co-disjunct nor co-alternate to q.	p is independent of q.
5. $(d, \bar{c})$: p is co-alternate but not co-disjunct to q.	p is sub-opponent to q.
6. $(c, \bar{d})$: p is co-disjunct but not co-alternate to q.	p is super-opponent to q.
7. (c, d): p is co-disjunct and co-alternate to q.	p is co-opponent to q.

Here the symmetry indicated by the prefixes, co-, super-, sub-, is brought out by reading downwards and upwards to the middle line representing independence. In this order the propositional forms range from the supreme degree of consistency to the supreme degree of opponency, as regards the relation of p to q. In traditional logic the seven forms of relation are known respectively by the names equipollent, superaltern, subaltern, independent, sub-contrary, contrary, contradictory. This latter terminology, however, is properly used to express the *formal* relations of implication and opposition, whereas the terminology which I have adopted will apply indifferently both for formal and for *material* relations.

CHAPTER IV

SECONDARY PROPOSITIONS AND MODALITY

§ 1. The division of propositions into simple and compound is to be distinguished from another division to which we shall now turn, namely that into primary and secondary. A secondary proposition is one which predicates some characteristic of a primary proposition. While it is unnecessary to give a separate definition of a primary proposition, a tertiary proposition may be defined as one which predicates a certain characteristic of a secondary proposition, just as a secondary proposition predicates some characteristic of a primary proposition. Theoretically this succession of propositions of higher and higher order could be carried on indefinitely. But it should be observed that any adjective that can be predicated of a *primary* proposition can be significantly predicated of a proposition *as such*, i.e. equally of a primary, a secondary, and a tertiary, etc. proposition; and that, in consequence, although *propositions* may be ranged into higher and higher orders, adjectives predicable of propositions are of only one order, and will be called "pre-propositional." Taking p to stand for any proposition we may construct such secondary propositions as: p is true, p is false, p is certainly true, p is experientially certified, p has been maintained by Berkeley. Here we are predicating various adjectives (the precise meaning of which will be considered later)

of any given proposition p; and we define each of these propositions—of which the subject-term is a proposition and the predicate-term an appropriate adjective—as secondary. One example may be given which has historic interest. Take (A) the proposition 'two straight lines cannot enclose a space'—to illustrate a primary proposition; again take (B) the proposition 'A is established by experience' as a secondary proposition; and thirdly take (C) 'B is held by Mill' as a tertiary proposition; namely—'It is held by Mill that the theorem that two straight lines cannot enclose a space is established solely by experience.' It is at once obvious that, all these three propositions, the primary, the secondary, the tertiary, which include the same matter (viz. that expressed in the primary) might be attacked or defended on totally distinct grounds. We may defend the primary proposition: 'two straight lines cannot enclose a space,' by showing perhaps that it is involved in the definition of 'straight'; again we might attack or defend the secondary proposition: 'this geometrical theorem is established by experience,' by considering the general nature of experience, and the possibilities of proving generalisations; and lastly, if we are to examine the tertiary proposition, namely 'Mill held the experiential view on the subject of this geometrical axiom,' we have only to read Mill's book and try, if possible, to understand what was the precise view that he wished to maintain.

§ 2. In connection with a larger and wider treatment of secondary propositions in general, it will be useful here to introduce the subject of Modality. We shall throughout speak of modal *adjectives*, instead of

modal *propositions*; it being understood that these adjectives fall under the general head of what we have called *pre-propositional* adjectives. We propose provisionally to include under modals the adjectives 'true' and 'false.' But a question of some interest arises as to whether the two very elementary cases 'p is true' and 'p is false' where p is a proposition are legitimate illustrations of secondary propositions. It may be held that the proposition 'p is true' is in general reducible to the simple proposition p; so that, if this were so, 'p is true' would only have the semblance of a secondary proposition, and would be equivalent for all ordinary purposes to the primary proposition p. It appears to me futile to enter into much controversy on this point, because it will be universally agreed that anyone who asserts the proposition p is implicitly committing himself to the assertion that p is true. And again the *consideration* of the proposition p is indistinguishable from the consideration of the proposition p as being true; or the attitude of *doubt* in regard to the proposition p simply means the attitude of doubt as regards p being true. These illustrations, in my view, show that we may say strictly that the adjective *true* is redundant as applied to the proposition p; which illustrates the principle, which I have put forward, that a proposition by itself is, in a certain sense, incomplete and requires to be supplemented by reference to the assertive attitude. Thus the assertion of p is equivalent to the assertion that p is true; though of course the *assertum* p is not the same as the *assertion* that p is true. The adjective *true* has thus an obvious analogy to the multiplier *one* in arithmetic: a number is unaltered when

multiplied by unity, and therefore in multiplication the factor *one* may be dropped; and in the same way the introduction of the adjective *true* may be dropped without altering the value or significance of the proposition taken as asserted or considered.

More interest attaches to the apparently secondary proposition 'p is false.' It certainly appears that p-false is indistinguishable from not-p, and the majority of logicians rather assume that not-p is on a level with p, and may be at once co-ordinated with p as a primary proposition. Now it appears to me that, while p-true is practically indistinguishable from the primary proposition p, on the other hand p-false is essentially a secondary proposition, and can only be co-ordinated with primary propositions after a certain change of attitude has been adopted. This problem will come up again in the general treatment of negation and obversion.

§ 3. We may now turn to what have been always known as modal adjectives such as necessary, contingent, possible, etc. The discussion of modality is complicated rather unfortunately owing to certain merely formal confusions which have not been explicitly recognised. Hence, before plunging into the really difficult philosophic problems, these formal confusions must be cleared away. The simplest of these occurs in the controversy between those who hold that contradictories belong to the same sphere of modality, and those who hold that they belong to opposite spheres of modality. This controversy is resolved by explicitly realising the distinction between a primary and a secondary proposition. Thus taking, for purposes of illustration, the antithesis between necessary and con-

tingent, we may consider the primary proposition 'It is raining now' and its contradictory 'It is not raining now'; if one of these primary propositions is contingent, so also is the other. But the contradictory of the secondary proposition affirming *contingency* of the primary—i.e. 'that it is raining now is contingent'—is the secondary proposition which affirms *necessity* of the primary—i.e. 'that it is raining now is necessary.' Thus, in doubting or contradicting a *secondary* proposition, we use the opposite or contrary modal predicate; but in denying the *primary* proposition we should attach the same modal adjective to the proposition and to its contradictory. There can really be no difference of opinion on this subject; the *opposition* of modality is expressed in the *secondary* propositions that contradict one another; the *agreement* in modality holds of the *primary* propositions that contradict one another. Summarising: if a given primary proposition is necessarily true, its contradictory, which is also a primary proposition, is necessarily false; and if a given primary proposition is contingently true, its contradictory, which is also a primary proposition, is contingently false. Thus in both cases the contradictory primary propositions belong to the *same* sphere of modality. But the contradictory of a secondary proposition affirming necessity or contingency of a primary, will be the secondary proposition which affirms contingency or necessity of the primary. Thus the contradictories of the secondary propositions assert *opposite* modals.

It is necessary to enter into the more philosophical aspect of modality, if only in a preliminary and introductory way, because, apart from the confusion between

a secondary and a primary proposition, there is, it would appear, considerable confusion in regard to the terminology adopted by different logicians or philosophers in their treatment of modals. To do this we feel bound to reconsider entirely the terminology. Since Kant it has been customary to make a three-fold division, using the terms apodictic, assertoric, and problematic; and this trichotomous division at once leads to some unfortunate confusions. The precise significance of assertoric in particular is peculiarly ambiguous: thus the proposition '2 and 3 make 5' as it stands, would appear to be merely assertoric; so that assertoric would include apodictic as one of its species[1]. Let us then begin our investigation without any bias derived from the traditional terminology.

§ 4. The first antithesis that immediately impresses us in this connection is that between a *certified* and an *uncertified* proposition. A proposition which is *uncertified* appears to be what Kant and others have sometimes meant by a problematic proposition; hence we begin by replacing the term 'problematic' by the term 'uncertified.' The contradictory of uncertified is certified, so that all propositions may be divided into the two exclusive classes of certified and uncertified. It is of course obvious that these terms are what is called relative; that is to say, at one stage in the acquisition of knowledge a given proposition may be uncertified, while at a later or higher stage, or with increased opportunity of observation, etc., it may become certified. The

[1] This confusion is, of course, due merely to the failure to distinguish between a primary proposition as such and a secondary. It is totally independent of any question as to what the adjectives *assertoric* and *apodictic mean* respectively.

distinction therefore is of course not permanent or absolute, but temporal and relative to individuals and their means of acquiring knowledge. It might be held that such distinctions should be excluded from Logic; but this, in our opinion, is unsound, in as much as reference to the mental powers and the individual opportunities of acquiring knowledge turns out in many discussions to be a most essential topic for logical treatment. The whole doctrine of probability hinges upon our realising the changeable or relative opportunities and means, which differ, from one situation to another, in the extent of attainable knowledge. The further discussion then of uncertified propositions will later introduce the logical topic of probability. Returning to certified propositions, a distinction is required according as the given proposition is certified as true or certified as false; and thus we have a triple division: uncertified, certified as true, and certified as false. But for most purposes this latter distinction is unnecessary, because for the given proposition that has been certified as false we might substitute the contradictory proposition that has been certified as true. It would be enough therefore to use the two divisions uncertified and certified, understanding by certified 'certified as true.'

§ 5. The above division leads to a fundamentally important subdivision under the term 'certified'; for we must recognise, in epistemology or general philosophy, that there are essentially different principles or modes by which the truth of a proposition may be certified; and a rough two-fold classification will conveniently introduce this subject: thus we may contrast a proposition whose truth is certified by pure thought or reason

with a proposition which is certified on the ground of actual experience. Briefly we shall call these two classes 'formally certified' and 'experientially certified.' The range of these two modes of certification will be a matter of dispute: some philosophers hold that all the principles and formulae of logic, and all those of arithmetic and mathematics, are to be regarded as certified by pure thought or reason. This gives perhaps the widest range for the propositions that may be said to be formally certified. But even amongst these, we may have to distinguish those which *have* been formally certified, from amongst the entire range which may be regarded as formally *certifiable*. Others would hold that many mathematical principles, such as those of geometry, can only be certified by an appeal to sense-perception—a form of experience; and thus the limits to be ascribed to the range of formal certification would open up serious controversy. Again, on the other hand, the range of propositions immediately certifiable *by experience* raises serious problems. Some may hold that the only truths guaranteed by mere experience are the characterisations of actual sense-impressions experienced by the thinker at the moment in which he asserts the proposition; many would extend this to judgments on the individual's past experiences revived in memory; but the most universally understood range of experientially certified propositions is still wider: it would include sense-perceptions, and observations of physical phenomena, and even judgments on mental phenomena,—these supplying the required data for science in general. We will not then profess to draw the line precisely between propositions that are to be regarded as formally certifiable and those

that are to be regarded as experientially certifiable; but there is one explanation of the relation between these two classes which will probably be admitted by all; namely, that propositions which are admittedly based on experience, will also involve processes of thought or reasoning, and that therefore no propositions of any importance are based upon experience *alone*; since an element of thought or reason enters into the certification of all such propositions. This leads to a simple, more precise definition of the antithesis—formal and experiential: while we define a formally certifiable proposition as one which can be certified by thought or reason alone, we do not define experiential propositions as those which can be certified by experience *alone*, but rather as those which can *only* be certified with the aid of experience. In this way we imply that experience alone would be inadequate[1].

A certain relation between the two antithetical modals, formal and experiential, will be found to apply over and over again to other antitheses in the characteristics of propositions. It may be illustrated by reference to the syllogism. Thus a certain syllogism may contain one formal premiss and one experiential premiss; and the conclusion deducible from these two premisses must be called experiential, because it has been certified by at least one experiential premiss. To put it otherwise, if all the premisses of an inference were formal,

[1] Even this distinction requires amendment; for it may be maintained that just as experience alone can certify nothing, so thought alone can certify nothing. Thus formal certification would coincide with what requires only *experience in general* (to use Kantian terminology) whilst experiential certification would involve in addition special or particular experience.

the conclusion would be formal; but if *only one* premiss is experiential (even though the others may be formal), the conclusion must be experiential. This particular characteristic of the syllogism is not arbitrary, but follows from the common understanding of what is meant by 'experientially certified,' namely something which could not be certified without experience,—*not* something which could be certified by experience *alone*.

§ 6. One of the chief sources of confusion is the use of the term 'necessary' in various different senses as an adjective predicable of propositions. It has sometimes been said that *all* propositions should be conceived as necessary; in the sense that the asserter of a proposition represents to himself an objective ground or reference to which he submits and which restrains the free exercise of his will in the act of judgment. This contention is indisputable, and may be regarded as one of the many ways in which the nature of judgment or assertion as such may be philosophically expounded. But obviously necessity as so conceived cannot serve as a predicate for distinguishing between propositions of different kinds. We pass, therefore, to the next and more usual meaning of the term necessary which will perhaps best be indicated by a quotation from Kant: 'Mathematical propositions are always judgments *à priori* and not empirical, because they carry with them the conception of necessity, which cannot be given by experience.' Here necessary is opposed to empirical; and the antithesis that Kant has in view coincides approximately with that between the *formally* certified and the *experientially* certified (as I have preferred to express it). But still another meaning has

been attached to the term necessary, viz., that according to which the necessary is opposed to the contingent. If, however, the term contingent is interpreted as equivalent to (what I have called) experientially certified, then we might agree that necessary should be interpreted as equivalent to formally certified; and in this case we should not have found a third meaning to the term. The question therefore arises whether a use can be found for the antithesis 'necessary' and 'contingent,' *within the sphere of the experientially certified.* Now it has been maintained as a fundamental philosophical postulate that 'All that happens is necessitated'; and this may be taken as equivalent to saying that 'Nothing that happens is contingent." It should here be pointed out that this contention is to be clearly distinguished from the view that 'All judgments or propositions are *necessary.*' For the necessity ascribed to judgments is conceived as a compulsion exercised by the objective or real upon the thinker; whereas the necessitation attributed to events is conceived (more or less metaphorically) as a compulsion exercised by nature as a unity upon natural phenomena as a plurality. The former necessity is so to speak objectivo-subjective; the latter objectivo-objective. But an elementary criticism must be directed against the use made of the postulate 'All that happens is necessitated' to deduce that there is no proper scope for the term contingent. For we inevitably conceive of that which happens as being necessitated *by something else that happens* in accordance with (what is popularly called) a law of nature. In other words, the laws of nature *taken alone* do not necessitate any event whatever; we should have rather to say that a law of

nature necessitates that the happening of some one thing should necessitate the happening of a certain other thing. Hence, I should propose that *nomic* (from *νόμος*, a law) should be substituted for necessary as contrasted with contingent. Thus a nomic proposition is one that expresses a pure law of nature; and a contingent proposition is one that expresses a concrete event. In this way we have eliminated the ambiguous term necessary, and have substituted *formally certified* when the term is opposed to experientially certified; and *nomic* when opposed to contingent. Finally the term *possible* must be coupled with the word *necessary* in its three usages. For 'possible' has three obviously distinct meanings: (1) what is not known to be false; i.e. what does not contradict the necessary in the first sense, applicable to *all* assertions; (2) what does not conflict with any formally certified proposition, i.e. with any proposition necessary in the second sense; (3) what does not conflict with any law of nature, i.e. with any proposition necessary in the third sense. The word 'possible' in these three senses may be distinguished respectively as 'the epistemically possible,' 'the formally possible' and 'the nomically possible.'

§ 7. It will now be apparent that the antithesis between nomic and contingent is of a totally different nature from that between certified and uncertified, or between the different modes of certification. The latter has been called subjective, the former objective; but the terms epistemic and constitutive are preferable: for the characteristics 'nomic' and 'contingent' apply within the content of the proposition, and are therefore properly to be regarded as constitutive; whereas the character-

istics 'certified' and 'uncertified' apply to the relation of the proposition to the thinker, and should therefore be called epistemic. Taking for example, the proposition 'Nature is uniform': if this is held to be necessary in the sense that our reason alone establishes its truth, then the attribution of necessity is in this case of the same kind as what we have called formally certified and is thus epistemic. But the necessity involved in the laws of nature is generally attributed to Nature itself, and not merely to our grounds for asserting such uniformity: and is thus constitutive. Thus, if we say, as a specific example of the necessity attributed to Nature's processes, that 'bodies attract one another in obedience to the necessities of nature,' this statement is quite independent of any view we may hold as to the reasonable grounds for asserting the fact of universal gravitation. In short, referring back to the distinction between the fact and the proposition, such modals as certified and uncertified are adjectives directly characterising the *proposition*, whereas modals of the other kind, typified by nomic and contingent, directly characterise the *fact*.

§ 8. It remains now to introduce a certain familiar distinction amongst propositions not included in the understood meaning of modal, viz. that between real and verbal. These terms were used by Mill, and are generally understood as equivalent to Kant's terms 'synthetic' and 'analytic.' Mill's point of view is very different from Kant's, for Mill is thinking of the nature of language, of the definition of words, etc., while Kant is thinking of ideas and the various constructive acts of thought. Mill's usage is more easy to expound than

Kant's, and gives rise to less serious conflict of view. A verbal proposition is one which can be affirmed from a mere knowledge of the meanings of words and their modes of combination; a real proposition, on the other hand, requires for its acceptance, not only a knowledge of the meanings of words, but also a knowledge of matters of fact. We may therefore note the same relation between verbal and real as between formal and experiential[1]: namely, that two premisses, both of which are verbal, can only yield a verbal conclusion; and that a single real premiss, even though joined with any number of verbal premisses, will impose upon the conclusion its own character as real.

The definition so far given of verbal propositions seems fairly clear; it is therefore surprising that it should have proved a stumbling-block to some logicians. The people who have raised difficulty on this point are those who have preferred the Kantian terms 'analytic' and 'synthetic' in place of Mill's terms 'verbal' and 'real': ('analytic' Kant illustrates by the proposition 'Material bodies are extended,' 'synthetic' by the proposition 'Material bodies attract one another'). The controversy has arisen through a tacit confusion between 'verbal or analytic' and 'familiar' on the one hand, and between 'real or synthetic' and 'unfamiliar' on the other hand, due to the kind of examples chosen to illustrate each type of proposition. This confusion is apparent in the well-known dictum of Bradley—'that synthetic judgments are analytic in the making'—where it is clear that by a 'synthetic judgment' he means the newly-constructed proposition, and by 'in the making,'

[1] See above, last paragraph of § 5.

the process of rendering the proposition familiar. But, it needs only a little reflection to show that familiarity with a matter of fact does not render the proposition which expresses such fact verbal or analytic; nor does unfamiliarity with the meanings of words render a proposition which explains such meaning real or synthetic: a proposition about the meanings of words is verbal, and a proposition about matters of fact is real, whether the hearer is unfamiliar or familiar with the words or with the facts[1]. Thus the proposition '7 and 5 make 12' is familiar enough, but whether or not it is verbal (or analytic) has absolutely nothing to do with its familiarity; on the other hand, a technical definition given by a scientist will probably be quite unfamiliar, but if the scientist puts it forward as an expression of his intention to use a word with a certain significance, the proposition which states his intention is verbal, although it is *ipso facto* unfamiliar.

Perhaps a better way of indicating the nature of a verbal proposition, is to say that it is not quite what is ordinarily meant by a proposition; that is, as verbal, it cannot strictly be said to be either true or false, because it does not declare a fact, but rather expresses an intention, a command, or a request. The technical scientist puts forward his definitions in this spirit, when he asks readers to allow him to use a term with a cer-

[1] An important explanation of all this should be given. What Bradley means by "an analytic judgment"—not "a verbal proposition"—is a judgment that could be discovered by introspective analysis, so that his pronouncement is an obvious truism. But it is strange that he does not perceive that this is not in the least the same as what Kant meant. Kant's distinction is epistemological, Bradley's merely psychological.

tain signification which is explained by his definition. Thus a verbal proposition is neither true nor false, because it is properly expressed, not in the indicative, but in the imperative or other similar mood. But if by a verbal proposition is meant one that assigns the meaning of a word as conventionally used in any wider or narrower context, then, inasmuch as the proposition asserts the fact that such or such *is* the convention, it must be either true or false.

§ 9. At the beginning of this chapter we defined a secondary proposition as one that predicates one or other of the adjectives significantly predicated of a proposition as such. We proceeded to consider in turn different kinds of adjectives that are thus predicable: this has led to a discussion of modal adjectives, and has included in particular a consideration of the adjectives true and false, and finally of the predicates 'verbal' or 'analytic' and 'real' or 'synthetic.'

CHAPTER V

NEGATION

§ 1. Under the general problem of the nature of negation we may begin by considering the particular form of negation which has been called 'pure negation.' There appear to be several different meanings attached to the notion of *pure* negation: it may mean the simple attitude of rejection, as opposed to that of acceptance, towards a proposition taken as a unit and without further analysis. Such negation may be called pure, because the negative element does not enter within the content of the assertum, but expresses merely a certain mental attitude to the proposition itself. According to this definition of pure negation, the judgment which may be called purely negative has as its object precisely what I have called a secondary proposition in my previous discussion as to whether the statement '*p* is false' is to be regarded as primary or as secondary. When then we enquire as to the importance or the relevance of pure negation, we may be raising the question whether a judgment expressed in this purely negative form really ever represents a genuine attitude of thought. No doubt there are not many cases in which this negative attitude towards an assertum taken as a unit could be illustrated; but we may at least insist that, when some assertum is proposed which can be clearly conceived in thought, and yet repels any attempt to accept it, then the attitude

towards such an assertum to which our thinking process has led us is strictly to be called that of pure negation. For example, the proposition 'Matter exists' may appear to some philosophers to have in it a sufficiently clear content to enable them to reject it, without their having in mind any correspondingly clear substitute which they can accept. In this case their mental attitude towards the proposed assertum may be properly called one of mere negation; since the only positive element involved is the conceived content of the proposition rejected.

But the term pure negation is more generally applied where a predicate is denied of some subject *within* the proposition. Under this head, the case where negation would seem to be quite pure may be illustrated by a proposition like—'Wisdom is not blue.' Such a proposition would have purpose only in a logical context where we are pointing out that certain types of adjective cannot be predicated of certain types of substantive. A more common case which leads to a purely negative form of predication, is where, for instance, a distant object of perception, is considered as to whether it is blue or of some other colour, or as to whether it is a man or some other material body. Towards this proposed assertum—that it is blue, or that it is a man—our attitude may be that of mere denial, in the sense that we are perfectly clear what it is *not*, but we are not correspondingly clear as to what it is. We may admit that a judgment which in this sense is merely negative and without any positive content is rare, since when we deny of a flower that it is red, we are at least judging that it has some colour, and similarly when we deny of something in sight that it has the shape of a man we are

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at least judging that it has some shape, and this constitutes a positive element in our judgment. The above examples illustrate two applications of the notion of negation: first, in denying the proposition as a whole, and again in denying that an adjective of a certain type can be predicated of a certain type of substantive, where the positive element is evanescent; and secondly, in denying the more specific predicate proposed for a substantive while tacitly asserting some wider predicate under which it falls, where a positive element is properly to be recognised.

Some logicians, going one step further, have asserted that, in denying an object to be red, not only is the *generic* adjective colour a positive factor in the judgment, but that some specific colour other than red is tacitly affirmed: that is, they hold that we cannot deny unless we have some positive *determinate* ground for our denial. But this reason for asserting the universal presence of a positive factor in judgment must not be confused with the former; for it is one thing to say that the denying of any proposed adjective involves the affirming of *some* other adjective of the same generic kind, and another thing to say that it involves the affirming of a *specific* adjective. While admitting the first, I reject the view that in denying red we are affirming say green or blue as the case may be, on the ground that it involves a confusion between what is necessarily determined in *fact* with what may or may not be determinate in our *knowledge of fact*. There are countless cases of our denying a certain proposed adjective in which, while we know that *some* determinate adjective can be truly applied, yet we do not know

which determinate adjective is to be substituted for that rejected. The most obvious illustration is in predications of place and time: thus we may say 'Mr Smith is not now in this room,' and, knowing that Mr Smith is alive, we know that in the necessities of nature he must be in some other determinate place. Thus we may in a rapid survey discover the *absence* of any object within a given place, independently of any knowledge—by observation or otherwise—of its presence in some other place; and this is sufficient to dispose of the contention that there must be positive ground for a negative judgment. In fact the strictly negative form of judgment is relevant for purposes of further development of thought, whether we are able to assert an opposed positive, or know only that *some* opposed positive could be affirmed if our knowledge were further extended. What is obviously true of time or place predications is also, though not always so obviously, true of qualitative predicates such as colour or tone: for instance, we may deny that a certain sound is that of a piano, because of our familiarity with that instrument, without being able to define the kind of musical instrument from which the sound proceeds, owing perhaps to our unfamiliarity with other instruments; although we may know, first, that it is a musical sound, and secondly on quite general grounds that it must come—not from any instrument whatever—but from some determinate kind of instrument.

§ 2. Having distinguished some of the different ways in which the phrase pure negation may be understood, we will briefly examine the dictum that pure negation has no significance. It may perhaps be at once said that this dictum is itself purely negative, and that therefore

anyone who maintains its significance has committed himself to a contradiction. A more serious treatment of the contention shows that for the word 'significance' we should substitute 'having value' or 'importance' or 'relevance to a specific purpose.' The purpose, for instance, of the above negatively expressed dictum is to oppose some other philosophers who have attributed a false value or importance to the negative judgment. It will be seen that the whole question hinges on the meaning to be attached to the word 'significance.' A form of words may be said to be *absolutely* non-significant when they fail to convey any precise content for thought-construction. This failure of a phrase to convey meaning may be due either to the substantial components themselves or to their mode of combination; thus it is a merely *verbal expression* that may be said to have or not have significance for thought in this absolute sense. But in attributing non-significance to a judgment apart from its verbal expression, the most probable meaning intended is that it does not represent any actual process in thought. But any of the examples taken above go to show that the purely negative judgment cannot be universally charged with non-significance in this sense.

§ 3. We have considered in turn, first the proposition as a whole unanalysed; secondly, the predication of an adjective of a given subject-term; and we now turn to the subject-term itself, apart from the adjective predicated, and raise the question whether any proposition can have significance in case there is no real thing corresponding to the subject-term, although there may be a word or phrase used professedly to denote such thing. Now I have regarded the substantive, which is ultimately

the subject in all propositions, as a determinandum—that is, as something given to be determined in thought; if then there is *nothing* given to be so determined corresponding to the word or phrase by which we intend a certain substantive, then what becomes of the proposition? Consider for example the propositions: 'An integer between 3 and 4 is prime,' and again 'An integer between 3 and 4 is composite.' It must be said that neither of these propositions is true. Now since every integer is either prime or composite, it can be at once seen that any proposition predicating an adjective of the subject 'an integer between 3 and 4' must be false, even though the adjective is appropriate to integers as such. This statement needs only the qualification that we may correctly predicate of an integer between 3 and 4 that it is greater than 3 and less than 4; this, however, is not a genuine proposition but one that is implied in the meaning of the subject-term, and is thus merely verbal. We conclude then that of such a subject-term as 'an integer between 3 and 4' no adjective can be truly predicated in a real or genuine proposition.

We may therefore contrast two cases of a subject-term S: (1) where S is such that *some* adjective can be truly predicated of it in a genuine proposition, and (2) where S is such that *no* adjective can be truly predicated of it in a genuine proposition. These two cases may be briefly expressed—'S is' and 'S is not.' The significance of these two propositions is brought out in considering the process technically known as obversion. The fundamental problem of obversion I will symbolise as the problem of passing from 'S is-not P' to 'S is non-P.' Here, when we hyphen the negative with the

copula, I understand it to mean that the proposition '*S* is *P*' as a unit, is asserted to be false. But when we hyphen the negative with the predicate, we are affirming of the subject *S* the kind of predicate called negative; in other words '*S* is non-*P*' is an affirmative proposition containing a negative predicate, while '*S* is-not *P*' is a negative proposition in the sense that the attitude of negation applies to the proposition as a whole. Now this transformation from the negative proposition to the positive assertion of a negative predicate, has been assumed as almost trifling, and as only too obvious; but I would wish at once to raise the question as to the condition necessary for the validity of this process, called obversion, in its fundamental form.

As we have already stated, the incomplete proposition '*S* is' really means, '*S* denotes something of which some adjective may be predicated truly in a proposition not merely verbal.' Thus the scheme by which I express the condition under which obversion is valid, is to add to the explicit negative premiss '*S* is-not *P*,' the additional premiss '*S* is,' from which we may validly infer the affirmative conclusion '*S* is non-*P*.' The incomplete form of proposition '*S* is' means that *S* has some character which may be predicated of it, without defining what character can be positively asserted. The conclusion '*S* is non-*P*' means that we predicate of *S* a character, determined so far as that it is an opponent of the proposed character *P*, but otherwise indeterminate. An illustration from history will show how this process may be applied. Thus the name *William Tell* is the name of a historical character about whose existence there appears to be doubt. In denying

any proposition which predicates an adjective such as 'submissive' of the subject William Tell, we could not validly predicate of him the contrary adjective 'defiant,' unless we were able first to assert that Tell *is*, in the sense we have explained.

The problem of the *obversion* of a singular proposition is the same as that of formulating accurately the *contradictory* of a singular proposition. Thus, in showing that, in order to pass from the denial (or contradictory) of 'S is P' to the affirmation 'S is non-P' we require the additional datum 'S is,' we have indicated that neither of the propositions 'S is P' and 'S is non-P' would be true, in the case that 'S is' were not true. In other words, the two propositions 'S is P' and 'S is non-P' are not properly contradictories. The contradictory of 'S is P' should be formulated in the alternative proposition 'Either S is-not or S is non-P'; as also the contradictory of 'S is non-P' in the alternative proposition 'Either S is-not or S is P.' Thus, in our historical illustration, *neither* of the two propositions 'A certain man named William Tell *submitted to* the Austrians' and 'A certain man named William Tell *defied* the Austrians' would be true, if it were the case that there was no such person as William Tell; and hence the proper contradictories of the two propositions must be respectively expressed in the alternative forms: 'Either there was no such person as Tell or he (Tell) defied the Austrians,' and 'Either there was no such person as Tell or he (Tell) submitted to the Austrians.'

§ 4. To illustrate the significance of this view we must consider the different types of cases in which a proposition of the form—'S is'—can be truly asserted.

In every case, the term *S* must have sufficiently determinate meaning, to give rise to the alternative propositions '*S* is' or '*S* is not'; the question could not arise if *S* were treated as a mere symbol without significance. When this is agreed, it will be found that any apparent variations in the meaning of the word 'is,' will in reality be variations in the kinds of substantive category to which the name *S* is understood to apply. For instance, let us take the names of substantives under the category of number. We may say on the positive side that the number 3 *is*. This will mean that some true adjectives can be predicated of the number 3, beyond those which might be held as merely involved in the definition or connotation of the word 3; thus, if we should define 3 as meaning $2+1$, the statement that the number 3 has the characteristic expressed by $2+1$ would be purely verbal. But the number 3, we say, is such that an indefinite number of other adjectives, not included in its definition, can be truly predicated, as for instance that 3 is prime or that 3 is a factor of 12. Contrast the name 3 with the phrase 'an integer between 4 and 5': in the sense in which we can significantly assert that 3 *is*, we may assert that an integer between 4 and 5 *is not*; in other words, no true character can be assigned to this proposed subject, except what is involved in our understanding of its meaning, namely that it belongs to the general category of integer, and that it is to be greater than 4 and less than 5. Generalising from this example, it will be seen that such a subject-term is defined first by reference to a general category (in the above case that of number) and next, by a proposed means of determining or

selecting out of the members of that category, a particular example related in a defined way to other things.

With reference to the category to which the subject-term S by definition belongs, any difference of category is naturally associated with an apparent difference in the meaning of 'is.' In particular there is a range of subjects for which the word 'exists' would be naturally substituted for 'is.' Thus it may be agreed that what is manifested in space and time may be said to exist: hence we raise such questions as whether God exists, or whether the centaur Cheiron existed, or whether William Tell existed. The objects intended to be denoted by these subject-terms may be said to belong to the category of the *existent* whether the propositions asserting their existence are true or false. Thus we must maintain, in accordance with the nature of the definition of God, that 'God is an existent,'—this being a merely verbal or analytic proposition; but the question of the truth of the synthetic or real proposition 'God exists' remains problematic. The same holds of Cheiron and William Tell. On the other hand what is denoted by such a subject-term as 3 or an integer between 4 and 5 would not be called an existent. Thus we maintain that there is no difference in the force of the word 'is' in its isolated usage; but that if any difference appears—as when we substitute 'exists' for 'is'—this is merely due to a difference in the category of the subject-term, which again presupposes a difference in the types of adjectives that are properly predicable of it.

When the proposition 'S is' is under consideration it must be understood that the term S is not an ordinary singular name but one of a peculiar nature that has not,

I think, been recognised by logicians. Such a name will be designated by the prefix 'a certain.' Consider the following propositions: 'A certain man was both a philosopher and a historian,' 'a certain integer between 3 and 11 is prime,' 'a certain novel has no hero,' 'a certain flash of lightning was vivid.' The truth or falsity of these propositions could only be decided by the hearer if for the phrase 'a certain' is substituted 'some or other,' 'one or more,' so that for him the reference is indeterminate. Thus, of Hume and of Xenophon it is true that they were both historians and philosophers; between 3 and 11 there are two numbers—5 and 7—that are prime; and the other examples are equally ambiguous. We must suppose that the speaker has in mind a single determinate philosopher-historian, number, novel or flash, which has been identified by him, and to which therefore he may return in thought. From these examples we see that a term may be properly called uniquely singular for the asserter, although in fact there may be several objects answering to its explicit description. Thus from a proposition with the predesignation 'a certain' may be inferred the corresponding proposition with the predesignation 'some or other,' though of course not conversely. This points to two modes in which what is technically called the particular proposition can be inferred: first, from premisses one of which is itself particular; and secondly, from a specific instance for which the predesignation 'a certain' stands.

Now it is the latter form of proposition which raises the problem of the significance of the proposition '*S* is.' For the asserter, the contradictory of the proposition that 'A certain man was both an historian and a philo-

sopher' would be that the person of whom he is thinking was not both an historian and a philosopher: or the contradictory of the proposition that a certain integer between 3 and 11 is prime, would be that that same integer is composite; whereas for the hearer, who can only understand the given propositions as particular, the contradictory in the first case would be: 'No man is both an historian and a philosopher' and in the second, 'No integer between 3 and 11 is prime.' In fact, in denying the proposition that a certain integer between 3 and 11 is prime, we must mentally specify the integer about which we are thinking, and assert that this integer is composite. The form of the statement thus reached is equivalent to that of the conclusion in the process of obversion, but it is not obtained here (as in obversion) by the medium of the purely negative premiss, but directly by mentally specifying the number under consideration.

§ 5. It remains to explain more precisely the nature of the denial of '*S* is *P*' which combined with '*S* is' yields the conclusion '*S* is non-*P*.' The proposition which merely denies '*S* is *P*' must be understood to involve a hypothetical element. Consider, for example, the statement 'Anyone who calls this afternoon is not to be admitted'; this proposition does not contain any categorical assumption that somebody will call, and may be otherwise expressed in an explicitly hypothetical form 'If anyone calls he is not to be admitted.' Combining this premiss with the further ascertainable fact that a certain person has called, the obvious conclusion, that this person is not to be admitted, follows. In general the symbols that we have used, namely that '*S* is *P*'

is false and that 'S is,' may be explained by making explicit the descriptive, adjectival, or connotative factor in the term symbolised by S, which factor we shall symbolise by M. The negative premiss then becomes: 'If anything is M it will not be P': the categorical premiss becomes: 'A certain thing is M.' In this formulation the symbol S does not appear. Now S stands for a certain thing which has not yet been identified and which is only presented in thought by the general description M; or briefly S is to mean 'a certain thing which is M.' In transforming the proposition 'Anything that is M will not be P' into the form 'S will not be P' we introduce the factor 'a certain thing.' And in transforming the proposition 'A certain given thing is M' into the form 'S is' we have transferred the whole of the adjectival component in the proposition from the predicate to the subject: or otherwise, the two propositions may be rendered 'Anything that may be given having the character M will not be P' and 'A certain thing having the character M is given.' Thus it is not strictly correct to use the same symbol S in our two propositions, since the only differentiating element of meaning in the term S in the negative premiss is the adjectival or descriptive component, whereas in the categorical premiss the substantival component enters along with the adjectival. It follows that the analysis given is not restricted to the negative form of our first premiss, since the same kind of syllogism would apply to an affirmative conclusion: the essential characteristic of the first premiss is its hypothetical character, as opposed to the other premiss which is categorical. Thus the affirmative case would

be rendered 'Anything that may be given having the character *M* will be *P*,' 'A certain thing with the character *M* is given' therefore 'This thing having the character *M* will be *P*.'

§ 6. In the course of this final explanation it will be noted that for the formula '*S* is' in which no determinate adjective is predicated, we might substitute '*S* is given' or '*S* is real.' Now the words 'given' and 'real' though of course grammatically adjectival, are not in the logical sense adjectival, for their meaning does not contain any indication of character or relation. It may be remarked in passing that the application of the term 'real' includes but goes beyond that of the word 'given.' The postulate that has to be assumed is that, however indeterminately we may have been able to characterise it, the real must have some determinate character. We thus return to our first exposition of the force of the incomplete predication '*S* is': namely that *S*, as being real, must have some determinate character although it may be that this character cannot be completely or exactly known by any finite intelligence.

CHAPTER VI

THE PROPER NAME AND THE ARTICLES

§ 1. A *simple* proposition '*S* is *P*' involves as subject a single uniquely determined substantive, and as predicate a single uncompounded adjective; a *singular* proposition must satisfy the first of these conditions, but its predicate may be simple or compound. Propositions of this nature give rise to the question how the reference in the subject *can* be uniquely determined. Speaking generally, singular names may be divided into two classes according as they contain or do not contain an explicit adjectival or relational component: to the first class belong such terms as 'the smallest planet,' 'the king of England who signed Magna Charta,' 'the cube root of 8'; to the second class 'Mercury,' 'John,' and '2'; and the former will, for convenience, be referred to as descriptive or significant, the latter as proper or non-significant.

Compare now the proper name 'Poincaré' with the descriptive name 'the President of France.' In order that this latter term may have unique application its component 'France' must have unique application: and hence here, as in almost every case, the uniqueness of a descriptive name is only secured through its reference to a proper name. On the other hand we shall find that many so-called proper names contain a descriptive factor: thus the term 'England' contains the termination 'land' which would be normally understood as bringing the

term 'England' within the general class expressed by the word 'land.' This, however, would not explicitly hold of the name 'France'; but, in point of fact, just as England might be taken to *mean* 'the land of the Angles,' so might France be taken to *mean* 'the land of the Franks.' In the same way the terms 'Mr Gladstone' and 'Lord Beaconsfield' have the partial significance expressed by the prefixes 'Mr' and 'Lord' respectively. As another example, the name 'Mont Blanc,' though etymologically equivalent to 'white mountain' and therefore apparently completely significant, must yet be called a proper name since its application is not to *any* white mountain, but to a specific one. We are here in the reverse position to that reached in discussing the term 'England'; for in 'England' we detected the concealed element of significance indicated by the termination 'land,' while in 'Mont Blanc' we have detected the concealed element of non-significance which prevents us from applying the name to *any* white mountain indiscriminately. Now, attributing to the term 'Mont Blanc' the maximum of significance that it can bear, and agreeing that there would be a species of incorrectness in using the term for any object which had not the characteristics 'white' and 'mountain,' yet this admitted significance is not the sufficient ground for applying the term as it is understood by those who use it with a common agreement as to its unique application. We may therefore say that any name which is commonly called a proper name has, so far as our analysis has proceeded, a residual element of non-significance over and above such significance as is naturally recognised in the verbal structure of the name.

§ 2. But if we allow of any name that it contains an element of non-significance, how is it possible that this name should be understood as applying to the same object when used at different times or by different persons or in different and varying connections? Where the name denotes a substantive, the possibility that it should mean the same substantive when used in different propositions, involves the possibility of substantival identification. A similar process, i.e. adjectival identification, is involved when the same adjective is used in different connections. Whether we ask how the substantive-name 'Snowdon,' for instance, can be understood to stand for a definite substantive, or how the adjective-name 'orange' can be understood to stand for a definite adjective, we are in fact confronted with precisely the same logical problem; and hence, if we regard the name 'Snowdon' as a proper substantive-name, we must regard 'orange' as a proper adjective-name. The analogy may be pressed a little further and applied to complex names; for just as an adjective name is exhibited as significant when it is expressed in the form of adjectives combined in certain relations, so a subtantive name is exhibited as significant when it is expressed in the form of substantives combined in certain relations. Note the analogy, for example, between the complex adjective-name 'the colour between red and yellow' and the complex substantive-name 'the highest mountain in Wales.' Here the former involves the proper adjective-names 'red' and 'yellow,' just as the latter involves the proper substantive-name 'Wales.' There is, then, in every explication of significance, a residual element in which we reach either a substantive-

name or an adjective-name which can no longer be defined in this form of analysis. And further, the explication of the significance of this residual proper substantive or proper adjective-name involves the conception of identity: in the one case, substantival identity—which is implied when we understand that the substantive denoted by the word 'Snowdon' in one proposition is identical with that denoted by 'Snowdon' in another proposition; and, in the other case, adjectival identity—which is implied when we understand that the adjective denoted by 'orange' in one proposition is identical with the adjective denoted by 'orange' in another proposition.

Our first approximate account of a proper name is then, that the intended application of the given name is to an object—whether it be substantive or adjective—which is identical with the object to which it may have been previously understood as applying in another proposition. For example, to explain what I mean by 'orange' I could say: 'You understand the word colour: and I shall mean by "orange" the colour which you can discern as characterising the object to which I am pointing. And when you identify the colour of any object with the colour of this, its colour is to be called "orange".' The possibility of such appeal presupposes that colour can be perceptually identified in different objects, apart from any other agreements or differences that the objects may manifest. In the same way the explication of a proper substantive-name requires a similar appeal, which assumes the possibility of identifying a concrete object when it may be presented or thought about in different contexts. Thus, if it was asked whom I meant when I talked of Mr Smith, I

might say: 'I mean the man to whom you were introduced yesterday in my study.' The agreement, therefore, which can be maintained in the application of a proper name amongst those who continue to use it with mutual understanding, is secured by what in a quite general way we may call the method of introduction. An object is introduced, and in the introduction a name is given, and when further reference is intended to the same object, the name is repeated which was given in the act of introduction. In this way the nature of the residual element or undefinable factor in adjectives and relations as well as that in the case of proper names is further explained.

§ 3. It is worth pausing here to point out a confusion frequently made in discussing the nature of the proper name. The confusion is that between the cause which has led people to choose one name rather than another *name* for a given application, with the reason for applying the name—once chosen—to one object rather than to another *object*. This confusion again can be paralleled in ordinary adjectival names as well as in substantival names: thus, it is one thing to assign the etymological causes of the use of the name 'indigo' rather than some other name to denote a particular colour, and another thing to assign the reason for applying the name, when it has once come into common usage, to one of the colours rather than to some other: the reason for this latter is that the colour presented in a given instance is identical with that to which the name indigo was originally given. This is exactly parallel to the ground on which we should justify our applying the name 'Roger Tichborne' to the man presented in court:

namely, the presumed identity of the man before us with the man to whom his godparents had given the name. Why they chose that name rather than some other name is a matter for historical enquiry. But, to repeat, the etymological or historical account of a name must not for a moment be confused with its significance, or with what, in the case of proper names (substantival or adjectival), takes the place of significance as the condition of mutual understanding.

§ 4. This discussion is closely bound up with the different ways in which the articles, indefinite and definite, are used. The indefinite article in its most general and completely indeterminate meaning is illustrated by such assertions as 'A man must have been in this room,' 'We need a sweep,' 'You ought to make a move with your bishop.' Now if we compare this use of the article with its meaning when it occurs at the beginning of a narrative as for instance: 'Once upon a time there was a boy who bought a beanstalk,' we note an important difference in its significance. In the first set of examples the full significance of the article is made explicit by substituting 'some or other': e.g. 'Some or other man must have been in this room'; in the case of a narrative, where the article prepares the way for future references to period, person or place, it means—not 'some or other'—but 'a certain.' Indeed our story might more logically have begun 'At a certain time a certain boy bought a beanstalk.' When the indefinite article is used in this way to introduce some period, person or place not otherwise indicated, it will henceforward be called the *Introductory* Indefinite, to distinguish it from the *Alternative* Indefinite.

Now suppose the narrative to continue: 'This boy was very lazy'; the phrase 'this boy' means 'the boy just mentioned,' the same boy as was *introduced* to us by means of the *indefinite* article. Here the article 'this,' or the analogous article 'the,' is used in what may be called its *referential* sense. The linguistic condition necessary to render such reference definite is that only *one* object of the class (whether person, period, or place) should have been immediately before mentioned. Other variations of the Referential Definite are such phrases as 'the former' and 'the latter,' which may be required to secure definite reference. The above analysis brings out the necessarily mutual association of the *introductory* use of the *indefinite* article with the *referential* use of the *definite* article. Again, instead of beginning the second sentence with the phrase 'this boy,' language permits us to use a pronoun: thus the word 'he,' in general, is sufficient to denote a specific individual understood by the verbal context; so that here the pronoun serves precisely the same logical function as the referential definite article 'the' or 'this.' A still more important further development of the referential 'the' comes up for consideration when, instead of depending upon immediacy of context—as in the preceding cases of 'this' and 'he'—we refer to an historical personage who has a wide circle of acquaintance as (e.g.) 'The well-known sceptical philosopher of the eighteenth century.' Here the phrase 'the well-known' functions as a referential definite, though there may have been no immediately previous mention of Hume, it being assumed that a certain philosopher will be unambiguously suggested to readers in general, in spite of the fact that there may have been

more than one person answering to the description 'sceptical philosopher of the eighteenth century.' This extended use of the referential definite is quite interestingly illustrated in Greek, where a proper name is prefixed by the definite article 'ὁ'; a usage which appears very happily to bring out the precise function of the proper name, as referring back to an individual who was originally introduced in history or otherwise under that name. The same holds in English of geographical proper names, e.g. the Thames, the Hellespont, the Alps, the Isle of Wight, etc. Lastly, in a narrative, the juxtaposition of a *proper name* with the introductory indefinite supplies a substitute for the referential definite. Thus our story about the beanstalk which begins with the introductory indefinite 'a boy' may be continued either by using the phrase 'this boy'—involving the referential article—or by the pronoun 'he'; or thirdly by the proper name which prepares the way for repeated reference to the same boy: 'Once upon a time there was a boy named Jack who bought a beanstalk.' It will be noted therefore that the way in which the proper name occurs in a narrative where it secures continuity of reference, illustrates the same principle as its use in ordinary intercourse, where it ensures agreement amongst different persons as to its single definite application: in both cases, the understanding of the application of the name involves reference back to the act of introduction, when the name was originally imposed.

There is an important analogy between the singular descriptive name of the kind illustrated by 'the well-known sceptical philosopher of the eighteenth century' and the proper name, in that frequently it is only within

a narrower or wider range of context that the proper name may be said to have a uniquely determined application. Thus, within a family, the name 'John' may be understood to denote one brother of that name; whereas, in a certain period in English history, it will denote the king who signed Magna Charta. That uniqueness of reference is relative to a particular context is similarly seen in such phrases as 'the table,' 'the garden,' 'the river,' which though applicable to different objects in different contexts are understood within a given circle or in a given situation to have a uniquely determined application. The article 'the' used in such cases may be called Indefinite Definite, to distinguish it from the most definite of all uses of the article, namely where the unique application is understood without any limitation of context—in cases, for example, like 'the sun,' 'the earth.'

We have thus divided articles (and what are logically equivalent to articles) into four classes: (1) the Indefinite Indefinite, otherwise the Alternative Indefinite; (2) the Definite Indefinite, otherwise the Instantial Indefinite, best expressed by the phrase 'a certain,' which includes the Introductory Indefinite; (3) the Indefinite Definite, otherwise the Contextual Definite, which includes the Referential Definite; and (4) the Definite Definite, for which the understood reference is independent of context.

§ 5. A special form of the contextual definite which is to be distinguished from the referential, is expressed by the terms 'this' and 'that' when used as demonstratives. Literally, the demonstrative method is limited to the act of introducing an object within the scope of

perception. But, when we point with the finger, for instance, to a particular person or mountain or star, our attempt to direct the attention of the hearer to the object intended may or may not succeed: if successful, it will be because there is no other conspicuous object belonging to the class indicated by the use of the general significant name (person, mountain, star, as the case may be) within the range of space to which we have directed attention. The condition for securing unambiguity is not that there should be only one object of the specified class within the range indicated, but that there should be only one such *visible* object; and here observe a parallel between the demonstrative definite, and the case illustrated by the example 'the well-known sceptical philosopher of the eighteenth century.'

§ 6. At this point in our discussion let us consider the special difficulty which attaches to the notion of a proper name. This problem presents a dilemma. If we maintain that the proper name is non-significant in some sense, then it would follow that any propositional phrase that might contain the proper name would be non-significant in the same sense. If, on the other hand, we attempt to assign some definite significance to the proper name, this will entail our substituting a uniquely descriptive name as equivalent in meaning to the proper name, in which case the distinction between the descriptive name and the proper name would vanish.

This problem raises a question relating to the wider problem of the definition of words or phrases. Taking the two words 'valour' and 'courage,' the brief formula 'valour means courage' is seen on reflection to be imperfectly expressed. Everybody would agree that what

is intended here is that the two terms *valour* and *courage* have the same meaning; i.e. that the quality meant by the one term is the same as the quality meant by the other. Hence a more correct expression than 'valour means courage' would be 'the word valour means what is meant by the word courage.' Where a phrase instead of a single word is under consideration the same principle is involved. For example, 'p is a factor of q' means-what-is-meant-by 'q is divisible by p'; or again 'some benefactor of A' means-what-is-meant-by 'one or other person who has benefited A.' These illustrations bring out the distinction between (*a*) the relation which one word or phrase may bear to another word or phrase, and (*b*) the relation which a word or phrase may bear to what is called its 'meaning.'

Now the propositions which allow us to substitute one phrase for another may be called bi-verbal definitions[1]; and the relation that is to be affirmed as holding between two such phrases must be expressed in the complex form 'means what is meant by,' or even—when we distinguish between the phrase which has not been understood and that which has been understood—in the still more complicated form 'is to be understood to mean what has been understood to be meant by.' This last complication brings out the purpose that a definition has always to serve; namely the elucidation of a phrase assumed to require explanation in terms of a phrase presumed to be understood.

[1] It has been suggested that a more correct substitute for '*bi-verbal definition*' would be '*translation*.' But whichever terminology is employed, the distinction between the kind of definition called *translation* and some more ultimate definition remains.

This formulation of the bi-verbal definition leads us to consider what, in contrast, we shall call the uni-verbal definition. When we speak of a phrase as being 'already understood,' it is equivalent to saying that the *meaning* of the phrase is known. The formula that 'phrase *p* means what is meant by phrase *q*,' in short, raises the question, What is it that phrase *q* means? Let us first consider the *kind* of entity that a phrase could mean. Phrases which would be grammatically or logically distinguished in regard to type or category would *mean* entities belonging to correspondingly different types or categories: thus one phrase would mean a certain proposition; another would mean a certain adjective, another a certain substantive, and so on. Thus 'courage' means a certain adjective or quality-of-conduct, 'horse' means a certain substantive or kind-of-animal. A phrase prefixed by an article such as a, the, some, every, any, requires special consideration. Thus, if we were to substitute for such phrase any phrase that means what is meant by the given phrase, the article or some equivalent would still remain. Thus 'the first novel of which Scott was the author' means what is meant by 'the romance that was written by Scott before any other of his romances.' In this bi-verbal substitution the word 'the' is retained. Now consider, in contrast to the proposition stating the equivalence in meaning of the above phrases, the proposition 'The first novel written by Scott was called *Waverley*'; or, inasmuch as there is only one novel that is known bearing this name, we may put the statement in the form: 'The first novel written by Scott was the novel called *Waverley*.' Such a proposition is of nearly the same

type as 'The author of *Waverley* was the author of *Marmion.*' In both of these propositions the relation of identity is asserted in regard to two uniquely descriptive terms. But neither of these propositions is verbal; in neither case could we substitute for the relation of identity the expression 'means what is meant by.' Hence we are not identifying the *meaning* of the two phrases: i.e. we are not identifying what is meant by one phrase with what is meant by the other. What then is it that we are identifying? In the language of Mill we should say we are identifying what is *denoted*—and in the language of Frege what is indicated or (as we prefer to say) factually indicated—by the one phrase with what is denoted or factually indicated by the other. Now, as our term suggests, an appeal to *fact* is required in order to understand what it is that is *factually indicated* in distinction from what is *meant* by a certain phrase. Hence, though a knowledge of the usage of language alone is sufficient to know what a phrase means, a knowledge of something more than mere linguistic usage is required to know what a phrase denotes or factually indicates, whenever we are dealing with a phrase that indicates something different from what it means. The word 'courage' or the phrase 'not flinching from danger' is of such a nature that there is no distinction between what it means and what it indicates or denotes; it is only phrases prefixed by an article or similar term for which the distinction between meaning and indication arises. Turn now to the peculiarities of the illustration given above: 'The first novel written by Scott was the novel called *Waverley*.' The interpretation of this statement is that the object indicated

by the phrase that stands first is the same as that indicated by the phrase 'the novel called *Waverley*' although the meanings of the two phrases differ. Take a parallel case: 'The colour of the object at which I am pointing is identical with the colour that is called red'; here again the identity of what is *indicated* by the two phrases does not carry with it identity in what is *meant* by the two phrases. In short, where we have an identification of what is indicated in spite of non-identity in what is meant, we recognise that the statement of identity is not merely verbal but factual.

In the above illustrations we have taken such names as Scott and Waverley to exemplify names universally recognised as proper; while the phrase 'the first novel written by Scott'—or any phrase having the same meaning—would be called descriptive in a sense primarily intended as antithetical to proper. Now one step was taken to bridge this antithesis when we used the proper name in the extended phrase 'the novel called *Waverley*': i.e. the single name Waverley is a proper name and the compound phrase 'the novel called *Waverley*' is constructed in the form of a descriptive name. We are thus leading up to the view that what is *indicated* by the descriptive phrase—'the novel called *Waverley*'—is identical with what is *meant* by the proper name 'Waverley.' Thus, in interpreting the simple proposition '*Waverley* was the first novel written by Scott,' which is recognised at once to be factual not verbal, we are identifying what is factually indicated by the subject and predicate terms respectively; and in the case of the proper name 'Waverley,' what it factually indicates is indistinguishable from what it means. Hence it seems

legitimate or possible to define a proper name as a name which *means* the same as what it *factually indicates.*

§7. We may now introduce the technical term 'ostensive' which will suggest as its opposite the familiar term 'intensive.' A proper name may be said to be ostensively definable in contrast to those more ordinary terms which are said to be intensively definable. This ostensive definition will be only a special instance of a form of definition involving the complex relation 'means what is indicated by'—a relation which is involved in any attempt to define a proper name by means of a descriptive name. The particular force of the notion of ostensive definition will now be explained, and it will be found to apply both to an adjectival and to a substantival name. The ordinary proper name applies to an object whose existence extends over some period of time and generally throughout some region of space. The appearance of such an object in perception (or rather of some spatially or temporally limited part of that object) provides the necessary condition for imposing a name in the act of indicating, presenting or introducing the object to which the name is to apply, and this it is that constitutes ostensive definition. In extending the notion of a proper name to certain adjectives our justification is that ultimately a simple adjective-name—such as red—cannot be defined analytically but only ostensively. Theoretically, we must suppose that any name, singular or general, proper or descriptive, substantival or adjectival, has originally been imposed on a particular occasion by a particular person or group of persons. In the case of ostensively defined names, the occasion on which definition is pos-

sible must be one on which the object is actually presented. When, however, the meaning or application of such a name has afterwards to be explained, or so-to-speak redefined, the only direct method is to secure for the enquirer another presentation of the object in question. Thus John Smith, having been presented to his family at birth—which we may take to be the occasion on which the name was imposed—must be presented again to the person ignorant of its application. Hence, in introducing a man under the name John Smith, we are using the same ostensive method as was required in the original definition, but such mention of the name does not, properly speaking, constitute definition. We are stating, in effect, the proposition—which is not merely verbal—that 'the person introduced is identical with the person upon whom the name was originally imposed.' This case of an ordinary substantival proper name is analogous to that of an adjectival name—say cochineal—which originally could only have been ostensively defined, and which must therefore be ostensively redefined for the person ignorant of its application, in the form 'the colour of this presented object is identical with that upon which the name cochineal was originally imposed'—a statement which again is not merely verbal. When a proper name is called arbitrary, this arbitrariness attaches only to the original act of imposition; but, when the application of the name is afterwards explained, such explanation is no longer arbitrary, since to be correct the *real* proposition that the substantive or adjective presented is identical with that upon which the name was originally imposed, must hold good, and this statement may be either true or

false, apart from linguistic convention. Furthermore, when ostensive definition is employed, it must be observed that we do not say that the proper name means what is *meant* by such a phrase as 'the object to which I am pointing' (which after all is only an instance of a descriptive phrase), but we say that the proper name means what is *indicated* by the descriptive phrase 'the object to which I am pointing.' For it is obvious in this case, as in the more general account of a descriptive phrase, that however we may further explicate the *meaning* of the phrase 'the object to which I am pointing,' the substituted phrase would not have the nature of a proper name but necessarily of a descriptive name.

When, then, finally we agree with the general position of the best logicians that the proper name (as Mill says) is non-connotative, this does not amount to saying that the proper name is non-significant or has no meaning; rather we find, negatively, that the proper name does not mean the same as anything that could be *meant* by a descriptive or connotative phrase; and positively, that it does precisely *mean* what could be *indicated* by some appropriate descriptive phrase. This exposition holds both for the names of objects which can be presented and thus ostensively defined; and also for the names of objects removed in time or place, for the definition of which a descriptive phrase (which is other than ostensive) must be employed.

CHAPTER VII

GENERAL NAMES; DEFINITION AND ANALYSIS

§ 1. HAVING, in the preceding chapter, distinguished the different kinds of articles, we now turn to a common characteristic in the use of an article, namely, its attachment to a *general* name. The general name has usually been differentiated by reference to number, and roughly defined as a name predicable of more than one object. In fact, however, there are general names such as 'integer between 3 and 4' or 'snake in Ireland' that are predicable of *no* object, while 'integer between 3 and 5' and 'pole-star' are general names predicable of only *one* object. There is therefore nothing in the *meaning* of a general name which could determine the number of objects to which it is applicable. Rejecting this reference to number, we may point out that a universal characteristic of the general name is its connection with the article—the use of the grammatical term 'article' being extended to include this, that, some, every, any, etc. All terms of this kind serve to determine the intended application of reference in a proposition, and hence might more properly be called *applicatives* or *selectives.* Now a general name is distinguished as that to which any applicative can be significantly prefixed: thus the applicative, on the one hand, requires a general name, while, on the other hand, it follows from the essence of the general name that to it any applicative is

significantly attachable. And a further and connected characteristic of the general name is that it can always be used in the plural, or, in fact, with any numerical prefix.

If now we further consider the typical applicatives 'every' and 'some,' we find that in the last analysis they entail the conception of unlimited generality, and language reflects this unlimited reference by uniting these applicatives with the substantival term 'thing' to form the single words 'everything' and 'something,' where the word 'thing' stands only for abstract generality without any such limitation as could possibly be defined by adjectival characterisation. If any account of the name 'thing' here could be given in philosophical terminology, we should have to say that 'thing' stands for the category of all categories; or more precisely, that the generality implied in this use of the word 'thing' is so absolute that it must include all logical categories, such as proposition, adjective, substantive, etc. On the other hand, the general term with which ordinary logic is almost exclusively concerned, is removed by two steps from this absolute generality: for example, the general term 'man' comes under the logical category 'substantive,' which itself is a limitation of the absolute generality peculiar to the word 'thing.' Thus in philosophical logic we ought to be prepared to define the category 'substantive' by some limiting characterisation within the absolute general 'thing'; and similarly of the category 'adjective' or 'proposition,' etc. And again, in ordinary logic, any general name is understood to be defined first, as coming under an understood category—in particular the category sub-

stantive—and secondly as delimited by a certain adjectival characterisation. As regards any specified general substantive-name, its nature as general is brought out by showing its meaning to be resolvable into some specific adjective or conjunction of adjectives, juxtaposed to the category 'substantive' itself.

The consideration that to the general name any applicative can be prefixed, distinguishes it from the singular name, whether descriptive or proper. We have now to bring out a characteristic shared in common by the general name and the singular descriptive name, which distinguishes them both from the proper name: namely that the two former always (except in the case of the absolute general name 'thing') contain in their meaning an adjectival factor, whereas the strictly proper name contains in its meaning no adjectival factor. This account may seem a somewhat arbitrary way of settling the prolonged controversy initiated by Mill as to whether proper names are non-connotative. In default of a definition of a proper name, however, it is impossible to decide whether any given name, such as London, is to be called proper. I propose, therefore, to define the word proper as equivalent to non-connotative, non-descriptive or non-significant (since these three terms are themselves synonymous), and the only debatable point which remains is as to whether any names can properly be called 'proper[1].' A similar remark applies to the question whether all general names are connotative, since it would seem necessary to define a general name

[1] This brief account of the Proper Name has been discussed and qualified in the preceding chapter; but for the present purpose it is adequate.

(except the absolutely general name 'thing') as equivalent to one which is connotative.

§ 2. At this point we must enquire into the precise meaning of the word *connotation*, and this enquiry necessitates the introduction of the more general notions of intension and extension. While extension stands for a set of substantives, intension stands for a set of adjectives; and moreover the two terms are used in correlation with one another, the substantives comprised in the extension being characterised by the adjectives comprised in the intension. Now of all the adjectives that may characterise each of a set of substantives, a certain sub-set will have been used for determining the application of the general name, and the range of extension thereby determined constitutes the denotation of that name. Thus the specific function of connotation is that it is used to determine denotation; and hence any other adjectives that may characterise all the substantives comprised in the denotation, do not determine the denotation, but rather are determined by it. The entire and often innumerable conjunction of adjectives determined by the denotation has been called by Dr Keynes, the *Comprehension*. Thus Dr Keynes's exposition may be summed up in the statement that connotation in the first instance determines denotation, which in its turn determines comprehension. The controversy on the question of what adjectives should be included in the connotation of the general term, has arisen from the false supposition that the logician starts with a known range of denotation, and that with this datum he has to discover, amongst the known common characters peculiar to the members of the class, those which shall constitute the

connotation. But this is reversing the order of priority, since there is no means of delimiting the range of denotation of a term otherwise than by laying down a finite enumeration of adjectives which, taken together, constitute the test for applying the name. This test, which is provided by the connotation, may have remained unchanged, or have varied in the course of time, or again have been technically determined by science in such a way as to conflict more or less with common usage; but in every case what constitutes the connotation is invariably a conjunction of adjectives which, within these limits of time or context, is *used* to determine the application of the name.

We are indebted to Mill for an elaborate and complete treatment of connotative names, including the connection between connotation and definition; but the formula that the definition of a general name is the unfolding of its connotation, must be corrected by the added reference to the substantival element, since connotation by itself is purely adjectival. It is obviously impermissible, for instance, to substitute for the substantival name 'a man' a mere adjective 'human' or the abstract term 'humanity'; rather 'a man' should be defined as 'a human being' or 'a being characterised by the attribute humanity.' In other words, a substantive-name must be so defined as to show that it is substantival. The word 'being' here has the force of the word 'substantive,' so that pushed to its logical conclusion, 'a man' should be defined as 'a human substantive'; and it is the *adjective* '*human*' which alone may require further analysis in our definition, not the word 'being' or 'substantive.'

The same principle will apply to the definition of a general adjectival name, if it is admitted that certain adjectives and relations (which may be called secondary) can be properly predicated of other adjectives (which in this connection might be called primary). For example, in the proposition: 'the unpunctuality of his arrival was annoying,' we appear to be predicating of the primary quality or adjective represented by the abstract name 'unpunctuality' the further or secondary adjective 'annoying.' By a secondary adjective we mean an adjective of an adjective, and if the primary adjective is expressed grammatically as a substantive, the secondary adjective would be expressed grammatically as an adjective, but if the primary adjective retains its grammatical form as an adjective, then its secondary adjective is expressed by an adverb, which is logically equivalent to an adjective of an adjective. Thus the following series of propositions: '*A* is moving,' 'the movement of *A* is rapid,' 'the rapidity of the movement of *A* is surprising'—involve the primary adjective 'moving,' the secondary adjective 'rapid,' and what we must here call the tertiary adjective 'surprising.' When the primary adjective 'moving' retains its adjectival form, the secondary adjective predicated of it would be expressed as an adverb; thus '*A* is moving rapidly'; or when the secondary adjective 'rapid' retains its adjectival form, the tertiary adjective predicated of it is again expressed as an adverb, as in '*A*'s movement was surprisingly rapid.' These examples seem to confirm the view that adjectives can properly be predicated of adjectives as such. If this analysis is correct, we should expect to find certain general adjectival names (ana-

logous to general substantival names) whose meaning could be elicited in terms of the secondary adjectives implied in their use. Thus to take Mill's example, the name 'fault' which is predicable of such primary qualities or adjectives as 'laziness,' 'unpunctuality,' 'untidiness,' is predicable of such qualities of conduct on the ground that these are characterised by the further or secondary quality 'faultiness.' In this way the definition of a *general* adjectival name would be formulated in terms of the secondary adjective or conjunction of adjectives, constituting its connotation, juxtaposed to the category adjective itself.

§ 3. Turning now to another aspect of our topic, we shall consider the nature of definition only in the form of bi-verbal substitution, where the reference is restricted to words or phrases; in contradistinction to ideas or things, which some philosophers have undertaken to define. The question of the definition of words requires a wider treatment than that generally accorded to it in logic; for under the influence, I presume, of scholastic doctrine, definition has been tacitly restricted to the case of substantive terms, to which the traditional formula 'per differentiam et genus' is alone applicable. It seems, indeed, as if logicians had shrunk in terror from the task of defining other than substantive-terms, setting aside preposition words, conjunction words, pronominal words and even adjectives, as if these, as such, were outside the scope of logical definition. The problem of definition, it is clear, must extend to *any* word, however it may be classified by grammar. We must certainly come to some mutual understanding of the meanings of such words as 'and,' 'or,' 'if,' inasmuch as

these and many other such words have a meaning, the understanding of which is essential in the pursuit of logic. It is frequently impossible, however, to define a word taken in isolation, and in such cases, we must construct, as that which is to be directly defined, a verbal phrase containing the word. We should not perhaps go far wrong if in every such case we took the completed *propositional* phrase as that which is to be defined, although it is often possible of course to take only some part of a proposition and succeed in meeting the requirements. We repeat then that our problem is how to define a given verbal phrase; and the answer is to substitute for it another verbal phrase. This is the complete and quite universal account of the *procedure* of definition, which justifies our restriction of the topic to bi-verbal definition; its obvious *purpose* is fulfilled if the substituted phrase is understood. (Cp. preceding chapter.)

In this connection it is worth noting that, when what has to be defined is a verbal phrase rather than a single word, we may italicise—so to speak—that part of the phrase for which an explanation is asked. In such cases the remaining components of the phrase may be, and generally ought to be, repeated in the phrase constituting the definition. In symbols, let us say that the phrase '*abcd*' requires definition, where the components '*bc*' are combined in the whole '*abcd*' in such way that the combination '*abcd*' is not understood. Suppose the symbol '*apqd*' to represent our definition; then we shall have defined '*abcd*' (where the component '*bc*' required explanation), by the phrase '*apqd*' in which '*pq*' explicitly replaces '*bc*,' and is offered in expla-

nation. A definition such as that symbolised above is rejected in current logical text-books on the ground that it commits the sin of tautology; for it repeats verbatim the symbol '*ad*' in the definition given as explanation of '*abcd*.' But this mode of definition, so far from being a ground of condemnation, exactly answers in the most adequate sense the requirements. The more exactly we repeat in our definition the actual words and their form of combination, used in the phrase to be explained, the more precisely do we meet the demands for an explanation. Typical instances of this principle will occur to anyone who reflects on the subject. It follows that no general or merely formal criticism of a definition can be made by any logical rule; the question whether any proposed definition is good or not being entirely relative to the enquirer's knowledge and ignorance of meanings. This by no means precludes the possibility of definitions which would be generally useful, because any obscurity or ambiguity which one person might feel is likely to be felt by others; a little common sense is in general all that is necessary.

This account leads at once to one conclusion, which is perhaps tacitly understood by all logicians and philosophers; i.e., that, inasmuch as the only way to explain one verbal phrase is to substitute another, therefore no successive chain of explanatory phrases can serve the purpose of ultimate explanation, if that chain is endless. Hence perhaps the important point in the theory of explanatory definition is that it must stop. In other words, by a shorter or longer process, every definition must end with the indefinable. A certain misunderstanding as to what in logic is meant by the indefinable

must here be removed; for it has been frequently supposed that the indefinable means that which is admittedly not understood. But so far from meaning the 'not-understood,' the indefinable means that which *is* understood; and philosophy or logic may ultimately adopt a term as indefinable only where, because it is understood, it does not require a further process of definition. Philosophy must never stop with the indefinable, in the sense of reaching a component of thought expressed obscurely or with an admitted margin of doubt as to meaning. The indefinable does not therefore mean that which is presented as having no understood meaning, but that whose meaning is so directly and universally understood, that it would be mere intellectual dishonesty to ask for further definition.

§ 4. There has been practical unanimity in regarding definition as a process which essentially involves analysis. We have above reached the idea of an indefinable; and it has been almost universal, I believe, to regard the indefinable as equivalent to what is incapable of analysis—a view which is obvious if definition and analysis coincide. But, if definition is not to be merely equated to analysis, then we may pause before we regard anything as being indefinable on the ground of its being, in some proper sense, unanalysable. In my own view, definition assumes so many varied forms that its equivalence to analysis seems to be highly dubious.

Before entering into further detail on this point it will be well to consider what is meant by the word *analysis* in philosophy and logic. Associated with it we often find a reference to parts and wholes: thus, analysis is

often said to mean the separation of a whole into its parts. But 'separation' does not adequately represent the process. For instance, grammatical analysis does not mean taking the several words of a sentence and determining the part of speech and inflection, etc., of each word, but its object is to show how the significance of the sentence is determined by the mode in which the several words are combined. Similarly the analysis of a psychosis is not merely the cataloguing of such elements as knowing, feeling, acting, but rather the representation of the essential nature of a given psychosis as determined by the mode in which these factors combine. What holds of grammatical and psychological analysis holds of every kind of analysis; and, since the important process is—not the mere revelation of the parts contained—but rather the indication of their mode of combination within the whole, analysis is better defined as the exhibition of a given object in the form of a synthesis of parts into a whole. In this way we can say that any process of analysis can also be described as a process of synthesis; but this does not amount to saying that analysis means the same as synthesis, any more than that the relation 'grandfather' is the same as the relation 'grandson,' although the fact that A is the grandfather of B is the same fact as that B is the grandson of A. In short, analysis is the inverse of synthesis; i.e. when the whole X is analysed into its several components a, b, c, d; then a, b, c, d have to X the inverse relation which X has to a, b, c and d. In this way it is clear that to analyse X simply means the same as to exhibit X as a synthesis.

Now instead of taking X as a term to be defined,

and exhibiting it as a synthesis of a, b, c and d, let us take the term a and define it by showing how it functions in a whole X where it is combined with b, c and d. This points to two modes of definition, viz. analytic and synthetic. In analytic definition we pass from an unanalysed—i.e. an apparently simple—to an analysed equivalent; in synthetic definition we exhibit the nature of what is simple, not by representing it as a complex, but by bringing it into synthetic connection in a complex of which it is a component. Or more shortly: analytic definition is explaining a complex in terms of its components, and synthetic definition is explaining components in terms of a complex. A few illustrations will help to make clear exactly what we mean by synthetic definition. Take, in arithmetic, the definition of 'factor' or 'multiple.' We first construct a certain complex, involving integers illustratively symbolised by a, b, c;—namely, the proposition $a \times b = c$; this complex, *being understood*, is used to define the terms that require definition, and the definition assumes the following form: a is said to be a factor of c, or c is said to be a multiple of a, when the relation expressed in the proposition $a \times b = c$ holds. Here we do not resolve the meanings of the terms 'factor' or 'multiple' into their simple components of meaning, but define them by showing in what way they enter as components into the understood construct—viz. the particular equational proposition. Again taking the algebraic definition of 'logarithm': we begin by constructing a certain whole—expressed for convenience in terms of general illustrative symbols b, l, p,—viz. the proposition that 'b to the power l equals p.' This constructed complex—

the nature of which is presumed to be understood—is the medium in terms of which the notion of a logarithm will be defined as follows: 'l is the logarithm of p to the base b' means what is meant by the proposition: 'b to the power l equals p.' Again take the definition of 'sine': here, starting with an angle A bounded at the point O by the lines OX, OR, we make the following construction; taking some point P in OR, and dropping the perpendicular PM upon OX, then the sine of A means the ratio of MP to OP. The constructed complex here, namely of a certain right-angled triangle, has first to be indicated and understood, and by its means we are enabled to define what may be called a component of this complex. Another example of this form of definition is afforded in any attempt to define the words 'substantive' and 'adjective.' Here we may presuppose that the notion of 'proposition' is understood—e.g. as that of which 'true' or 'false' may be significantly predicated—and further presupposing that the notion of characterisation is understood, the first account of substantive and adjective will be that they are combined in a proposition in a mode expressible either in the form that 'a certain substantive is characterised by a certain adjective,' or that 'a certain adjective characterises a certain substantive.' Here we give at the same time the definitions of substantive and of adjective by showing how, as components in the whole, i.e. the proposition, they have to be combined[1].

§ 5. The general notions of analysis and synthesis

[1] *All* the definitions occurring in a symbolic system, whether Logical or Mathematical, should in my view be synthetic (in the above sense) and never analytic.

are often explained in terms of parts and whole; but these latter terms should be used for a process essentially different from analysis. At least three processes which are commonly confused are here to be carefully distinguished, viz., partition, resolution, and analysis proper; probably other more subtle variations must finally be recognised.

By *partition* is meant transforming what is first presented as a mere unit by exhibiting it in the form of a whole consisting of parts; it is perhaps more generally defined as the process of dividing a whole into its part; but it is of the first importance to point out that until a thing is presented as having parts, it cannot be said to be a whole. This conception of part and whole should be strictly limited to three types of cases: (1) to an aggregate, and to a number as the adjective of an aggregate; (2) to what occupies space and to the space which it occupies; (3) to what fills time and to the time filled. These three cases bring out the essential nature of the conception, viz., that the parts must always be conceived as homogeneous with one another and with the whole which they constitute; and further that a certain character called magnitude is predicable of any whole, the measure of which is equal to the sum of the magnitudes predicable of the parts[1].

Next, consider the term *resolution*. This is very shortly explained as the process of exhibiting a composite in terms of its components; but such a definition is open to the same kind of criticism as we have levelled

[1] No intensive or qualitative characteristic of an object can be regarded as a 'whole' of which a magnitude can be predicated by addition of the magnitudes of its 'parts.'

against the popular definition of partition, and by a parallel emendation we shall say that resolution means the exhibition of what is presented as simple in the form of a composite of which the components are assigned. An example of psychological interest is the resolution of a chord heard into its component notes, or again of a note into its component tones, where in either case the combination describes or accounts for the sound as heard. While, on Helmholtz's theory of auditory sensations, the physiological process here involved would be represented as a whole capable of partition, it remains none the less true that psychological apprehension presents the sound as a composite to be resolved.

Then thirdly, I should restrict the word analysis to a process which is distinctively logical, and which assumes its simplest form when we combine various adjectives as predicable of one and the same substantive, by means of the mere conjunction 'and.' A simple example will bring out the distinction between resolution and analysis. We have shown what is meant by resolving a note into its component tones. Now the character of the note is described under certain aspects, such as pitch, intensity and timbre, and this constitutes an *analysis* of its character. These three characteristics are predicated of the sound, not in the sense of resolving the sound into various component sounds, but in the sense of characterising the sound itself, whether it be composite or simple. Thus taking timbre for instance as one of the constituent characters, if a note contains three partial tones this would count as 3 in its resolution but 1 only in its analysis; if on the other hand the note were simple—i.e. contained only one

component tone—this would count as 1 in the resolution, while the analysis of this single note yields the same 3 characters (pitch, timbre and intensity) as that of the composite note. The value of this illustration is that it conclusively disposes of the assumption that a plurality of predications characterising an object depends at all upon its partition or resolution; that is, upon regarding the object either as a whole consisting of parts or as a composite resolvable into components.

So far we have considered that form of analysis which exhibits its object as a synthesis of constituents conjoined by the conjunction 'and,' and yielding what will be termed a compound. In contrast to a compound synthesis we must consider also what must be called a complex synthesis—namely one in which the material constituents are heterogeneous, including substantives of different kinds and adjectives of different order—monadic, diadic, triadic[1]—which, *qua* heterogeneous, are united in different modes from that of simple conjunction. Thus the word 'courageous' yields the complex synthesis 'not flinching from danger'; where the material constituents are 'danger' and 'flinching from,' of which the former is expressed substantivally and the latter as a diadic adjective. From this fairly simple example it will be seen that the possible forms of complexity that analysis may yield are inexhaustible. The discussion of this topic will be continued from a somewhat different aspect in a subsequent chapter on relations.

[1] See Chapter IX.

CHAPTER VIII

ENUMERATIONS AND CLASSES

§ 1. An enumeration is an assignment of certain items which may be said to be *comprised* in the enumeration. We attach therefore to an enumeration the conception of unity as applied to the whole along with plurality of the items comprised in this whole. In naming the items to be comprised in an enumeration as *a*, *b*, *c*, *d*, *e*, for instance, it will generally be implied that we shall not repeat any item previously named; also, that the order of assignment is indifferent. For the purposes of elementary illustration, we shall consider that *all* the items ultimately to be enumerated have a finitely assigned number (say) 12: which may be named respectively *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *k*, *l*, *m*, *n*. Such an enumeration might be called our enumerative *universe*. Thus taking any assigned enumeration *included in this universe*, we may speak of the *remainder* to this enumeration—by which will be meant the items comprised in the universe, but *not* comprised in the first assigned enumeration. The notion of *remainder* is therefore associated with the notion of *not*; although the two must be strictly distinguished. The remainder to an assigned enumeration is the simplest function of a single enumeration with which we shall be concerned. We next consider the typical functions of two enumerations—*E*, *F* say:

namely 'E into F,' and 'E with F.' By the former is meant the largest enumeration which is included both in E and F; by the latter, the smallest enumeration that includes both E and F. Thus, let E be [a, b, c, d, e, f] and let F be [d, e, f, g, h]; then 'E into F' will be [d, e, f]; and 'E with F' will be [a, b, c, d, e, f, g, h]. Anticipating elementary arithmetical notions, we may at once assert the generalisations: first, that the number for 'E into F' cannot be greater than the number for E or the number for F; and secondly, that the number for 'E with F' cannot be less than the number for E or the number for F. If E and F are identical, then in every sense of the word equals, E into $F = E$ with $F = E = F$. This represents one limiting case. If, on the other hand, some items comprised in E are comprised in F, the number for 'E with F' is less than the sum of the numbers for E and F respectively, and if there are no items comprised both in E and in F, then the number for 'E with F' will equal the sum of the numbers for E and F respectively, while the number for 'E into F' is zero. In the former case E and F would be said to be *not exclusive*, and, in the latter, *exclusive* of one another. For the general case:

The number for (E into F) + the number for (E with F)
= the number for E + the number for F.

For the purpose of further development, we will abbreviate the term remainder into R'. The *symbol* R', i.e., *remainder to*, and the prepositions *with*, *into* may be called 'operators,' because each indicates a certain operation to be performed upon one or upon two enumerations by means of which another related single enumeration of the same order is to be constructed.

The relation of the given operation to the enumeration which is to be constructed will be called the relation of *yielding*. Now, noting that R' $[a, b, c, d, e, f]$ yields $[g, h, k, l, m, n]$ and that R' $[d, e, f, g, h]$ yields $[a, b, c, k, l, m, n]$, we may illustrate the import of these operators in the eight following formulations where the symbol = stands for *yields*:

(1) $[a, b, c, d, e, f]$ into $[d, e, f, g, h] = [d, e, f]$
(2) $R'[a, b, c, d, e, f]$ into $R'[d, e, f, g, h] = [k, l, m, n]$
(3) $R'[a, b, c, d, e, f]$ into $[d, e, f, g, h] = [g, h]$
(4) $[a, b, c, d, e, f]$ into $R'[d, e, f, g, h] = [a, b, c]$

where (1) with (2) with (3) with (4) = the enumerative universe.

(5) $R'[a, b, c, d, e, f]$ with $R'[d, e, f, g, h] = [a, b, c, h, k, l, m, n]$
(6) $[a, b, c, d, e, f]$ with $[d, e, f, g, h] = [a, b, c, d, e, f, g, h]$
(7) $[a, b, c, d, e, f]$ with $R'[d, e, f, g, h] = [a, b, c, d, e, f, k, l, m, n]$
(8) $R'[a, b, c, d, e, f]$ with $[d, e, f, g, h] = [d, e, f, g, h, k, l, m, n]$

where (5) into (6) into (7) into (8) = the enumerative zero.

As regards these eight formulae we observe that each of the pairs (1) and (5), (2) and (6), (3) and (7), and (4) and (8) give two enumerations related the one to the other as remainder. What holds in one illustration can be formulated in general terms. Let E and F be any two enumerations, then: the operation 'E into F' yields-the-enumeration-yielded-by the operation R' ($R'E$ with $R'F$). Since the relation 'remainder to' as also the relation 'yields-what-is-yielded-by' are reversible or symmetrical, this single formula includes all the eight formulae which have been illustrated above. But, we may for the sake of emphasis, express the principle again in eight formulations where = will now stand for 'yields-what-is-yielded-by.'

(1)	E into F	=the remainder to		$R'E$ with $R'F$
(2)	$R'E$ into $R'F$	= ,,	,,	,,	E with F
(3)	$R'E$ into F	= ,,	,,	,,	E with $R'F$
(4)	E into $R'F$	= ,,	,,	,,	$R'E$ with F

where (1) with (2) with (3) with (4)=the enumerative universe.

(5)	$R'E$ with $R'F$	= ,,	,,	,,	E into F
(6)	E with F	= ,,	,,	,,	$R'E$ into $R'F$
(7)	E with $R'F$	= ,,	,,	,,	$R'E$ into F
(8)	$R'E$ with F	= ,,	,,	,,	E into $R'F$

where (5) into (6) into (7) into (8)=the enumerative zero.

From any one of the above eight formulae we can read off any other. Where any one mode of constructing an enumeration is equivalent in the above sense of = to a certain other mode of constructing an enumeration, it is obvious that the equivalence will imply equality of number, although the reverse does not hold: that is, we may have equality of number for two enumerations while the items comprised in them are not necessarily the same.

§ 2. We have now to consider how a single enumeration may be taken as an item to be enumerated along with other enumerations so as to constitute an *enumeration of enumerations*, that is an enumeration comprising, as its items, units which are themselves enumerations. This conception of an enumeration *comprising* enumerations must not be confused with an enumeration *including* enumerations. Thus: if 'F includes E' then the items comprised in E are the same as some of the items comprised in F, and here E and F comprise the same types or kinds of items. But if 'F comprises E,' then the items comprised in F will be of a higher order or type than the items comprised in E. Using the term item (in the first instance) to stand for an entity of order

zero, i.e., one which is not itself an enumeration, an enumeration comprising such items will be of the first order, and an enumeration comprising enumerations of the first order will be of the second order; and so on. In passing from enumerations of the first order, viz., those which comprise mere items, to enumerations of the second order which comprise enumerations of items, we may symbolise the distinction by using square brackets. Thus an enumeration of the first order may be illustrated thus: $[a, b, c, d, e, f, g, h, k, l, m, n]$ involving one size of bracket. Now, with these same twelve items we may illustrate several enumerations of the second order which will involve *two* sizes of brackets as follows:

$$\Big[[a, b], [c, d, e, f, g], [h, k, l, m], [n]\Big]$$

where the item-enumerations are exclusive and four in number.

Again:

$$\Big[[a, b, c], [c, d, e, f], [a, f, h, k, l], [d, m, n]\Big]$$

where the item-enumerations are not exclusive of one another and again are four in number.

Or again:

$$\Big[[a, b], [c, d, e], [b, f, h], [d, k], [e, m, n]\Big]$$

where some pairs of the item-enumerations are exclusive and others not, the total number being five.

In all these illustrations, a comma is used to separate the items to be enumerated in constituting an enumeration, and the square brackets are used where required to indicate what is to be taken as an item. Similarly,

using the same twelve items we may illustrate an enumeration of the third order; which will involve three sizes of brackets thus:

$$\Big[\big[[a, b], [c, d, e]\big], \big[[b, f, h], [d, k]\big], \big[[e, m, n]\big]\Big],$$

where there are three items which are enumerations of the second order, of which the first two comprises as items two first order enumerations, while the third comprises only one first order enumeration. Comparing our illustration of a first-order enumeration with the first illustration of a second-order enumeration, we must note that the item n which is of zero order is to be distinguished from the item $[n]$ which is of the first order, and comparing this last with our third order enumeration, we must distinguish the item $[e, m, n]$ which is of the first order from $\big[[e, m, n]\big]$ which is of the second order. The distinction between n and $[n]$ is, therefore, that the former is to count as one along with other items in constituting an enumeration of the *first* order, while $[n]$ is to count as one along with other items in constituting an enumeration of the *second* order. Similarly, the distinction between $[e, m, n]$ and $\big[[e, m, n]\big]$, is that the former is to count as one along with other items in constituting an enumeration of the *second* order, while the latter is to count as one along with other items in constituting an enumeration of the *third* order. On precisely similar grounds we must distinguish between, say, $\big[[a, b], [c, d, e]\big]$ and $\Big[\big[[a, b], [c, d, e]\big]\Big]$; for the former represents an enumeration of the second order

comprising *two* items which are enumerations of the first order, while the latter represents an enumeration of the third order comprising *one* item of the second order. In the above treatment we have in effect defined the notions 'item' and 'enumeration,' not as having absolute significance, but as having relative significance, in the sense that the two notions are indicated by the relative term 'comprising' and its correlative 'comprised in.' In other words our proper topic has been the development of the kind of relation expressed by the verb *comprise*.

Further to illustrate the principle that the operators *into* and *with* yield an enumeration of the same order as the enumerations operated upon, we will apply these operators to enumerations of the second order. When abbreviating the expression for an enumeration by substituting a single letter E or F, we shall use as indices 1, 2, 3, ... to indicate the different *orders* to which any enumeration may belong. Thus:

Let E^2 stand for $\big[[a, b, c], [c, d], [a, e, f, g], [a, h, k, l]\big]$

and F^2 stand for $\big[[a, b, c], [c, d, e], [a, e, f, g], [b, k, l, m]\big]$

then the operation 'E^2 into F^2' yields $\big[[a, b, c], [a, e, f, g]\big]$

and the operation 'E^2 with F^2' yields

$$\big[[a, b, c], [c, d], [c, d, e], [a, e, f, g], [a, h, k, l], [b, k, l, m]\big]$$

Thus the operation E^2 into F^2 yields G^2 and the operation E^2 with F^2 yields H^2, where it may be seen that G^2 and H^2 stand for enumerations of the second order. We shall also require a symbol for the result of using the operator *into*, where the enumerations are exclusive

of one another. According as the enumeration yielded in this case is of the first, second, or third order, it will be symbolised by $[0]$, or $\big[[0]\big]$ or $\Big[\big[[0]\big]\Big]$. Thus:

The operation $[a, b, c]$ into $[d, e]$ yields $[0]$; and the operation $\big[[a, b, c], [c, d]\big]$ into $\big[[a, d, e], [c, f]\big]$ yields $\big[[0]\big]$. Thus amongst all possible enumerations of the first order we must include $[0]$; and amongst all those of the second order we must include $\big[[0]\big]$. An enumeration characterised as being 0 may be called an empty enumeration, the symbol 0 having in every case the same meaning: the single, double, etc., bracket *adds* a further character to the character zero. The symbol 0 is obviously selected to indicate that the number of items in an empty enumeration is zero. As regards items of zero order none can be called empty; and therefore none can be symbolised as 0.

§ 3. At this point we must explain more precisely the distinction between being comprised in and being included in. Thus, each of the three items a, b, c is comprised in the enumeration $[a, b, c]$, and no others. The relation of comprising thus always correlates an item or an enumeration of a certain order with an enumeration of the *next higher* order. On the other hand $[a]$ is included in $[a, b, c]$, thus showing that the relation of inclusion is a relation between enumerations of the *same* order.

The above account suggests the elementary problem, how many enumerations are included in the enumeration $[a, b, c]$ (taken to be an enumeration of the first order).

Since [o] being of the first order is included in *every* first order enumeration, the following first order enumerations will exhaust all those that are included in [a, b, c], namely:

First, that which comprises no item: [o]:

Secondly, those which comprise one item:

[a] and [b] and [c]:

Thirdly, those which comprise two items:

[a, b] and [a, c] and [b, c]:

Fourthly, that which comprises three items: [a, b, c]. Thus, within the enumeration [a, b, c] the number of distinct enumerations included is 8; for, in selecting any enumeration which shall be included in [a, b, c], we have, with respect to each of these three items, the *two* alternatives of admitting or omitting it. Hence the number of our choices is $2 \times 2 \times 2$. Similarly in any enumeration comprising (say) n items, 2^n enumerations will be included. Thus, we may write down an enumeration of the second order which shall comprise all the enumerations of the first order included in [a, b, c]. Thus: $\big[[o], [a], [b], [c], [a, b], [a, c], [b, c], [a, b, c]\big]$. In general terms then: The enumeration of the second order, that shall comprise as its items all the enumerations of the first order included in a *given* enumeration of the first order comprising n items, will comprise 2^n items of the first order.

§ 4. Having treated of enumerations we may now consider the relation between an enumeration and a class. Whether a class may or may not be considered as an enumeration of a special kind, it will be agreed that there is involved in the notion of a class an element

entirely absent from that of a mere enumeration. In the language of Mill, the denotation of a class may be said to be determined by connotation; i.e. by a certain conjunction of adjectives. But here it is of the utmost importance to note that, on the one hand, the substantival items constituting the denotation are united merely by the *enumerative* 'and'; but, on the other hand, the adjectival items constituting the connotation are united by the *conjunctional* 'and[1].' In fact, what is common to every logician's employment of the term *class* is that its limits are determined—not merely, if at all, by a mere enumeration of items—but essentially by the character or conjunction of characters that can be truly predicated of this and of that item that is to be comprised. The distinction between an enumeration and a class is closely connected with that between an extensional and an intensional point of view, so that logicians have contrasted the extension of a class with its intension or the intensional conception of a class with its extensional conception. Phrases of this kind have in fact been introduced in various parts of this work; but a more direct way of attacking our problem would be to speak —not of the extension and the intension of a class— but of the extension of an intension and of the intension of an extension, where the preposition *of* requires to be logically defined. More explicitly I propose to speak of an intension as determining a certain extension, or conversely of an extension as being determined by a certain intension. Thus the relation *determining* and its correlative *determined-by* will indicate the required connection and distinction. We shall not generally

[1] See Chapter III.

speak of an extension determining an intension or of an intension being determined-by an extension, but the relation of determining will be always from the intension to the extension. In short, it is this direction of determination which justifies Mill's use of the term *connotation*, and when we are conceiving the converse case of an extension determining an intension, then we may adopt for the intension in this case the convenient term *comprehension* as introduced by Dr Keynes. The word *determining* as used above is of course elliptical. In speaking of a given intension or conjunction of adjectives as determining an extension, what of course is always understood is that this or that item is or is not to be comprised in the extension according as it is or is not characterised by the given conjunction of adjectives. Now it will be found that the larger and more familiar part of logical theory is actually concerned —not with the notion of extension—but solely with that of intension, and that it is only when arithmetical predicates come into consideration that the notion of extension seems to be required. Thus, taking the proposition: 'Everything having the character m has the character p,' we may, for any English letter standing illustratively for an adjective, introduce the corresponding Greek letter in a purely *symbolic* sense to stand for the class determined by that adjective. Thus the above distributively expressed proposition may be rendered: 'the class μ is included in the class π.' Again, if we conjoin with the above proposition: 'Everything having the character p has the character m,' we reach the form of proposition: 'The class μ coincides with the class π.' Now the relation *coincides* is analogous to the relation

of co-implication, in that both are transitive, symmetrical and reflexive; i.e., they have the properties of equivalence or identity. In this way we may speak of the identity of a class determined by one adjective with that determined by another (merely as expressing a symbolic or abbreviated formula) without implying that there is any real entity to be called an extension or a class to which the strict relation of identity could be applied. All this is assumed in the next chapter, where we shall represent the force of propositions by means of closed figures. In spite then of the prominent employment of the word *class* both in the treatment of propositions and still more in that of the principles of syllogism, it may be maintained that there is no real reference in thought to the class as an extension, but only a figurative or metaphorical application of the word which serves to bring out certain analogies between such notions as inclusion, exclusion, and exhaustion which apply primarily to parts and wholes and are transferred as relations between propositions and their constituent elements. Some logicians have even gone so far as to say that the spatial relations amongst plane closed figures represent the actual mode of thought by means of which we are able to comprehend logical relations. I, however, reject this extreme point of view, but suggest that the mere fact that we are able to represent logical relations by analogy with relations amongst spatial figures almost justifies our maintaining that the idea of an extension determined by an intension is a logically valid concept. The full significance of such a scheme as Euler's diagrams for representing class-relationships has, in my view, been inadequately

recognised. It should be pointed out that the boundary line of a closed figure may be taken as the proper analogue of the intension, while the area within that boundary is the proper analogue of the extension. This suggestion brings out the following analogies: firstly, that it is intension which determines extension in the same way as a boundary line determines the enclosed area and separates this area from the remaining area outside; secondly, that we can apprehend in thought the full determining intension in the same way as we can optically grasp the single boundary in its entirety; and thirdly, that in general we cannot in thought enumerate all the items which are to be comprised in the extension, just as we cannot exhaustively present to the eye the several and innumerable points within the given enclosed area. On the other hand, though the several points cannot be exhaustively presented to the eye and yet the area presents itself ocularly as a unitary whole, similarly it would seem that though we cannot exhaustively enumerate in thought the members of a class yet we can conceive the class or rather the extension as a unitary whole. Again we may make within the area actual dots of a finite number which thus constitute a literal (though of course not exhaustive) enumeration, and thus the force of the diagram, as providing analogues to logical relations, is still further brought out, in that we may think one by one of the objects which we have selected—not arbitrarily—but on the ground that each of them is actually characterised by the adjectives which determine the class. Whether this analogy between a psychical image or perception of an area and the logical conception of a class, justifies our regarding the latter as

a genuine concept, is a debatable psychological problem. Dismissing this problem we must return to the strictly logical question whether a class is a genuine entity.

§ 5. If we provisionally allow a class comprised of individual existents to be an existent, then as an existent it is of a different order from any individual comprised in it. Similarly a class comprising adjectives—if it is to be called an adjective—must be an adjective of a different order from any adjective comprised in it. Similarly for a class of propositions. Therefore the question whether a class is a genuine entity admits at any rate that it is not of the same order of being as any item which it comprises. Whether this or that is a genuine entity can only be answered when we have provided a test of genuineness. The only general test which I can conceive of is as to whether the entity intended to be meant (in using such a word as *class*) can serve as subject of which some predicate can be truly asserted[1]. Thus, as an illustration of the *general* question, we may ask whether a proposition is a genuine entity, and taking the proposition *matter exists*, the reply would be in the affirmative, inasmuch as we can make the assertion 'that *matter exists* was rejected by Berkeley.'

Similarly with regard to the genuineness of a 'class' which is the topic under consideration. Taking for example the class *apostles*, we may assert 'that this class numbers twelve.' Inasmuch as this statement will be admitted to be true, the only relevant question that could arise would be as to whether the number *twelve* is predicated—*not* of the kind of entity called a class—but rather of its determining adjective. To this it may

[1] Cf. the treatment of '*S* is' in Chapter V.

be replied that since different intensions may determine the same class, the number predicated cannot be said to be predicated of one of these determining intensions rather than of any other. In other words, we may say that the adjective *twelve* is of such a kind that, taking any two co-implicative determining intensions, if it can be truly predicated of one it can be also truly predicated of the other. Since then it is indifferent of which of these several co-implicatives the adjective *twelve* is to be predicated, it seems to follow that it is not predicated of any of them whatever. Let us take p and q to stand for two co-implicative intensions. This means more precisely that, if anything is characterised as being p, it will be characterised also as being q; and conversely, if anything is characterised as being q, it will be also characterised as being p. It is then clear that the relation called co-implication is both symmetrical and transitive. Let us then assume the question at issue, namely that there is such an entity as a class. Then our conception of a class involves the universal statement that any given intension determines one and only one class. In this way the relation of co-implication subsisting between p and q, may be resolved into the statement that p determines a certain unique entity which is the same as that which is determined by q: that entity being what we have taken to be a class. Here the relation *determining*, which relates an intension to its extension, is what is called a many-one relation, because there may be many different intensions which determine a single extension. Similarly, the relation *determined-by*, which relates an extension to its intension, is what is called a one-many relation, because there is only one extension which is determined by many different intensions.

§ 6. We may generalise the above by taking, as a typical illustration, the relation of a *man* to the *country* which he inhabits. Since there is only one country which a man inhabits the relation *inhabits* is many-one, and its converse *inhabited-by* is one-many. Now let us combine these two correlatives in the following form: '*A* inhabits *the* country that is inhabited by *B*.' The relation thus constructed of *A* to *B* (which may be called the relation *compatriot-of*) is obviously symmetrical and transitive. It is symmetrical because there is only one country which *A* can inhabit and one country only which *B* can inhabit, so that to say that *A* and *B* inhabit the same country exhibits a symmetrical relation, since the terms *A* and *B* may be interchanged. Again, the relation is transitive since if *A* and *B* inhabit the same country, and *B* and *C* also inhabit the same country, it follows, since no one inhabits more than one country, that *A* and *C* inhabit the same country. The *identity* of the country inhabited by this, that and another man is a symmetrical and transitive relation, and it is upon these properties of identity that the symmetry and transitiveness of the relation *compatriot-of* depends. In fact, to speak of *the* country inhabited by *A* would not be a legitimate expression unless there was *one and only one* country which a man could be said to inhabit. Thus it will be seen that, when we combine in this sort of way any many-one relation with its correlative (which is necessarily one-many), then we have constructed a relation which has the two properties transitive and symmetrical. But the reverse of this does not obviously hold—that is, given a transitive and symmetrical relation it does not obviously follow that this relation can be resolved into a combination of a certain many-one

relation with its correlative. If, however, this converse proposition may be taken as axiomatic, it would follow, from the two properties—symmetrical and transitive—which hold of co-implication between two intensions p and q, that there is a certain thing which p determines and which is determined by q. To this kind of entity we apply the name *class*. It is therefore only by *assuming* the theorem that any given relation that is symmetrical and transitive can be resolved as above in terms of a many-one relation and its converse that we can *prove* that the notion of a class represents a genuine entity. But, in making use of the analogy between (a) 'the country inhabited by any of the men that are compatriots of one another' and (b) 'the class determined by any of the intensions that are co-implicants of one another,' it must be further pointed out that, just as the 'country' under consideration is not the same as the compatriots taken in their totality, so the 'class' under consideration is not the same as the co-implicants taken in their totality. Since, then, the 'compatriots' or the 'co-implicants' taken in their totality would constitute a *class*, the attempt to *prove* the above theorem would entail a *petitio principii*, when applied to the question of the genuineness of the notion 'class.' Merely from the symmetry and transitiveness of the relation ($\hat{s}$) compatriot, we cannot prove that there is a certain many-one relation ($\hat{r}$) inhabit, or (an entity x) viz. a certain country *distinct from the class of compatriots all of whom inhabit* ($\hat{r}$) *that country* (x). Our certainty that 'when a is $\hat{s}$ to b, then a is $\hat{r}$ to x and x is $\check{r}$ to b for some x' is due to the fact that, in order to construct any $\hat{s}$, we must first have been given $\hat{r}$.

CHAPTER IX

THE GENERAL PROPOSITION AND ITS IMMEDIATE IMPLICATIONS[1]

§ 1. We may define a general proposition as one in which the subject is constructed by prefixing an applicative to a general name. According to this definition, the only kind of proposition which is not general would be that in which the subject is expressed by a proper name; and the general proposition would include two forms of singular proposition: namely, where the general subject-term is prefixed by 'a certain' or by 'the.'

We must first point out that there are what may be called *pure* general propositions, where the general term is represented by the word 'thing' in its absolute universality: for example 'Everything is finite,' 'some things are extended.' In these propositions the subject-term has merely substantival without any adjectival significance. But ordinary propositions in every-day use apply to subjects adjectivally restricted; in other words, there is an adjectival significance in the subject-term as well as in the predicate term.

§ 2. Using the capital letters P and Q for general or class terms and the corresponding small letters p and q to represent their adjectival significance, then the correct

[1] This Chapter should be read in close connection with Chapter III.

expression for the affirmatives, universal and particular, would be:

Every P is q,

Some P is q,

which brings out the substantival significance in the subject-term and the adjectival significance of the predicate term. Now these forms are at once seen to be equivalent respectively to

A. Everything that is p is q,
I. Something that is p is q[1],

where the adjective p occurs as predicate in the subordinate clause, and the introduction of the word 'thing' indicates an ultimate reference to the absolutely general substantive. The *negative* general categoricals may first be expressed in the forms:

Everything that is p is-not q,
Something that is p is-not q,

where, in attaching the negative to the copula, it is to be understood that the adjective q is *denied*, in the first case, of everything that is p; and, in the second case, of something that is p. But these negatives are expressed less ambiguously as the contradictories of I and A respectively, thus:

E. Nothing that is p is q,
O. Not-everything that is p is q,

[1] The subject-terms in these two readings are often contrasted as being the one in denotation and the other in connotation. This disregards the fact that the meaning of the term P contains a connotative as well as a denotative factor, while the phrase 'thing that is p' contains a denotative as well as a connotative factor. Hence the two readings are not properly to be contrasted.

where the negative is prefixed to the propositions as a whole, and must not be falsely supposed to qualify the *subject-term.*

By expressing the propositions A, I, E, O, in the above forms the true logical nature both of obversion and of conversion can be explained. Thus the negative that is introduced or omitted by the process of obversion, is to be attached to the adjectival factor alone in the predicate; and hence the four propositions:

$$\left.\begin{array}{ll} A. & \text{Every} \\ E. & \text{No} \\ I. & \text{Some} \\ O. & \text{Not-every} \end{array}\right\} \text{-thing that is } p \text{ is } q$$

become (respectively) by obversion

$$\left.\begin{array}{ll} E_b. & \text{No} \\ A_b. & \text{Every} \\ O_b. & \text{Not-every} \\ I_b. & \text{Some} \end{array}\right\} \text{-thing that is } p \text{ is non-}q$$

where E_b, A_b, O_b, I_b, are of the *forms* E, A, O, I, but having in each case non-q for predicate in place of q. Here the suffix b indicates that the proposition is obtained by obversion. Applying again the process of obversion to E_b, A_b, O_b, I_b, we obtain

$$\left.\begin{array}{ll} A_{bb}. & \text{Every} \\ E_{bb}. & \text{No} \\ I_{bb}. & \text{Some} \\ O_{bb}. & \text{Not-every} \end{array}\right\} \text{-thing that is } p \text{ is non-non-}q.$$

Now, by the principle of double negation, non-non-q

is equivalent to q. Hence A_{bb}, E_{bb}, I_{bb}, O_{bb}, are respectively equivalent to A, E, I, O—showing that the propositions obtained by obversion are equipollent, i.e. formally coimplicant.

The process of simple conversion may be exhibited by interpolating intermediate steps, where first the adjectival factor in the subject is removed to the predicate; then, by the commutative law for conjunctives, the adjectives are transposed; and finally, the first adjectival factor in the predicate is removed back to the subject:

E. Nothing that is p is q
= Nothing is p and q
= Nothing is q and p
= Nothing that is q is p.

I. Something that is p is q
= Something is p and q
= Something is q and p
= Something that is q is p.

The forms A and O, not being directly convertible, must first be obverted and then converted, giving the contrapositive, i.e. the converted obverse, thus:

A. Everything that is p is q
= Nothing that is p is non-q
= Nothing that is non-q is p.

O. Not everything that is p is q
= Something that is p is non-q
= Something that is non-q is p.

We have so far taken E, 'No P is q' to be equivalent to 'Nothing is pq'; and I, 'Some P is q' to be equivalent to 'Something is pq.' In other words we

have reformulated the E and I propositions, in which the subject-term is restricted by the adjective p, by using as subject-term the unrestricted reference expressed by the word 'thing.' In the same way, A, 'Every P is q' and O, 'Not-every P is q' may be reformulated thus: A, 'Everything is $\bar{p}$ or q'; and O, 'Not-everything is $\bar{p}$ or q'; since '$\bar{p}$ or q' is equivalent to 'q if p.' In these reformulations, the form of the proposition as universal or particular and as negative or affirmative is unaltered: the E-proposition is expressed as an E-proposition, the I as an I, the A as an A, the O as an O. We have, in short, merely reduced all the propositions to the same common denominator (so to speak) by using the narrowest reference for our subject-term which will be sufficiently wide to include all the subject-terms that may occur in any given connected system of statements. This kind of transformation has been called 'existential,' but since the term 'existential' has been so persistently misunderstood, it will be preferable to speak of 'instantial' instead of 'existential' formulation. In this mode of formulation, a further question arises, namely that of *interpretation*: in particular, as to whether the proposition—given to be reformulated—is to be understood to include (implicitly or explicitly) the statement that there are instances characterised by p, where p is the adjective connoted by the subject-term: i.e. to include the affirmative statement 'Something is p.' Here it is to be observed that the instantial statement 'Something is p' is implicitly contained in 'Something is pq,' but *not* in 'Nothing is pq.'

§3. We will distinguish the proposition which *contains* the instantial affirmation of its subject-adjective from

the (otherwise) same proposition which does *not* contain this affirmation by using the suffix f for the former, and n for the latter. In order to change (say) T_n into T_f, where T stands for any proposition having an adjectival factor (say p) in its subject-term, we must *add to* T_n the statement 'Something is p,' i.e. we must *conjunctively* combine with T_n the instantial *affirmation*. And, in order to change (say) T_f into T_n, we must *subtract from* T_f the statement 'Something is p,' i.e. we must *alternatively* combine with T_f the instantial *denial*. Thus: given T_n, T_f will be expressed 'T_n *and* Something is p'; given T_f, T_n will be expressed 'T_f *or* Nothing is p.'

Applying these expressions to the forms E_n, I_f, A_n, O_f we have:

E_n	means	'Nothing is pq.'
I_f	„	'Something is pq.'
A_n	„	'Nothing is $p\bar{q}$.'
O_f	„	'Something is $p\bar{q}$.'

E_f	means	'Nothing is pq *and* Something is p.'
I_n	„	'Something is pq *or* Nothing is p.'
A_f	„	'Nothing is $p\bar{q}$ *and* Something is p.'
O_n	„	'Something is $p\bar{q}$ *or* Nothing is p.'

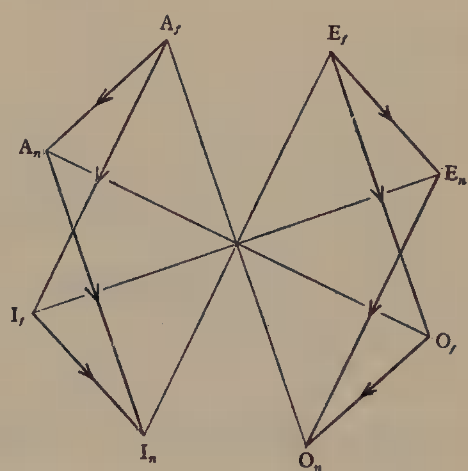
From the meanings of the symbols, as thus shown, the following rules will be evident:

(*a*) for obtaining the co-opponent: interchange f with n and A with O; or f with n and E with I (without any other change);

(*b*) for obtaining a sub-implicant; change either f into n, or A into I, or E into O (without any other change).

Since super-implication is the reverse of sub-implication, rule (*b*) reversed shows how to obtain a super-

implicant. Moreover, since a sub-opponent is a sub-implicant of the co-opponent and a super-opponent is a super-implicant of the co-opponent, rules (*a*) and (*b*) combined show how to obtain a sub-opponent or super-opponent. Lastly, since a sub-implicant of a sub-implicant is a sub-implicant, and a super-implicant of a super-implicant is a super-implicant, all the several sub-implicants, super-implicants, sub-opponents or super-opponents can be found. In the following mnemonic diagram, an arrow is used to point from a super-implicant to a sub-implicant, and co-opponents are placed diagonally opposite. The diagram is required in place of the ordinary square of implication and opposition, because of the distinction introduced between two possible interpretations of A, E, I, or O.



Here,

E_f, E_n, O_f are super-implicants of O_n, and are ∴ super-opponents of A_f,
A_f, A_n, I_f are super-implicants of I_n, and are ∴ super-opponents of E_f,
I_n, I_f, A_n are sub-implicants of A_f, and are ∴ sub-opponents of O_n,
O_n, O_f, E_n are sub-implicants of E_f, and are ∴ sub-opponents of I_n.

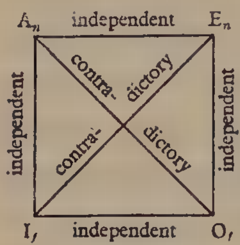
But, taking the laterally outstanding rectangle A_n, E_n, O_f, I_f, it must be observed that *no* relation of implication or opposition holds of A_n to E_n, of E_n to O_f, of O_f to I_f, or of I_f, to A_n; i.e. the sides of this rectangle exhibit the relation of *independence.*

The general nature of these results is that where any proposition is interpreted as having less determinate significance, it will be a super-implicant or super-opponent of fewer propositions and a sub-implicant or sub-opponent of a larger number. Thus A_f is super-implicant to A_n and I_f and I_n, but A_n is super-implicant only to I_n; and A_f is super-opponent to E_f and E_n and O_f, but A_n (being sub-opponent to O_n) is super-opponent to none of the propositions in the octagon. Conversely where any proposition is interpreted as having more determinate significance, it will be sub-implicant or sub-opponent to fewer propositions and super-implicant or super-opponent to a larger number.

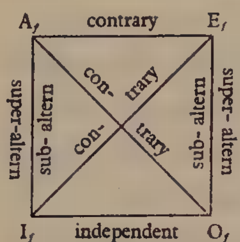
Similar modifications of the traditional scheme are required for inferences involving conversion. It will be found that equipollent conversion holds for E_n and I_f but not for E_f and I_n; and that sub-altern conversion holds in passing to I from A_f but not from A_n; and in passing from E to O_n but not to O_f.

Since each of the propositions A, E, I, O can be interpreted in two ways, there are several possible schemes of interpretation of the four together, in accordance with which a square can be extracted from the above octagon. Of these combinations, the following are the most reasonable.

(1) $A_n E_n I_f O_f$.

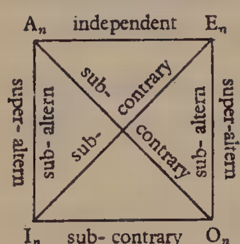


Here the universals are interpreted as containing no instantial affirmation, while the particulars implicitly contain instantial affirmation of the subject-term. This is the simplest interpretation, since each proposition is expressed as *uncompounded*, the universals being merely instantial denials, and the particulars, merely instantial affirmations.



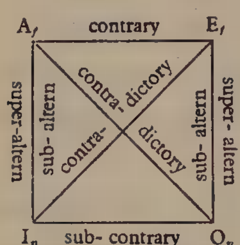
(2) $A_f E_f I_f O_f$.

Here all the four propositions are interpreted as containing instantial affirmation of the subject-term; so that the *universals* have to be expressed in a compound form.



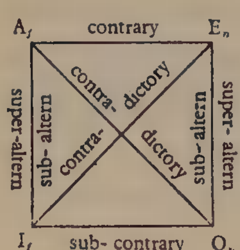
(3) $A_n E_n I_n O_n$.

Here all the four propositions are interpreted as containing no instantial affirmation, so that the *particulars* have to be expressed in a compound form.



(4) $A_f E_f I_n O_n$.

This is the reverse of the first interpretation, each proposition being expressed as a compound: the universals containing instantial affirmation of the subject-term, and the particulars containing no instantial affirmation.



(5) $A_f E_n I_f O_n$.

Here the affirmatives contain instantial affirmation of the subject-term, and the negatives contain no instantial affirmation.

§ 4. These five selected schemes must be compared with the traditional doctrine on the relations of A, E, I, O. Special explanation is required to justify the application of the terms contrary, contradictory, etc., in the traditional scheme, where all the four general propositions are to be understood to assume that there are instances of the subject-term. This interpretation differs from interpretation (2) given above, where each proposition is interpreted as containing this affirmation; for, in case there were no instances of the subject-term, either of the four forms of proposition would, on the traditional scheme be *meaningless*, whereas on interpretation (2) they would be *false*. Thus I_f and O_f would both be false, i.e. A_n and E_n would both be true, supposing there to be no instances of the subject-term; whereas the traditional scheme, precluding this possibility from the outset, asserts that I and O cannot both be false, i.e. that A and E cannot both be true. This presupposition of traditional logic is concealed from the ordinary reader by the universal employment of Euler's diagrams, in which the subject-class is indicated by an actual circle, so that the limiting case, where the class vanishes, is never represented. Furthermore, the conversions authorised on the traditional scheme derive their validity from the assumption that there are instances not only of p, but also of q and of non-p and of non-q, where p and q are the adjectival factors in the subject and predicate terms. These assumptions again are tacitly involved in Euler's diagrams, where the circles for P and Q are not allowed to vanish or to exhaust the universe.

§ 5. In what follows we shall adopt the traditional view that there are instances of p, of non-p, of q, and of

non-q; and on this assumption we proceed to consider all the formal relations amongst the propositions involving p or non-p with q or non-q. The symbols A', E', I', O' will stand respectively for the propositions A, E, I, O when modified by negating both the subject and the predicate adjective. Thus, using the following abbreviative substitutions, viz., 'All' for 'every,' 'p' for 'thing that is p,' and '$\bar{p}$' for non-p, we have

$A =$ All p is q	$A' =$ All $\bar{p}$ is $\bar{q}$
$E =$ No p is q	$E' =$ No $\bar{p}$ is $\bar{q}$
$I =$ Some p is q	$I' =$ Some $\bar{p}$ is $\bar{q}$
$O =$ Not all p is q	$O' =$ Not all $\bar{p}$ is $\bar{q}$

giving a list of eight *distinct* (i.e. non-equipollent) general categoricals, as an extension of the usual four. Adding to this list the propositions whose subject and predicate terms are p-$\bar{q}$, $\bar{p}$-q, q-$\bar{p}$, $\bar{q}$-p, q-p, $\bar{q}$-$\bar{p}$ we obtain in all 32 categoricals; of non-equipollents, however, there are only $32 \div 4$, since by means of obversion and simple conversion, each proposition may be expressed in four *equipollent* forms. This is shown in

TABLE I

	By Obversion (i)	⟷	By Conversion (ii)	⟷	(iii)	By Obversion ⟷	(iv)
A	All p is q	=	No p is $\bar{q}$	=	No $\bar{q}$ is p	=	All $\bar{q}$ is $\bar{p}$
A'	All $\bar{p}$ is $\bar{q}$	=	No $\bar{p}$ is q	=	No q is $\bar{p}$	=	All q is p
E	All p is $\bar{q}$	=	No p is q	=	No q is p	=	All q is $\bar{p}$
E'	All $\bar{p}$ is q	=	No $\bar{p}$ is $\bar{q}$	=	No $\bar{q}$ is $\bar{p}$	=	All $\bar{q}$ is p
O	Not all p is q	=	Some p is $\bar{q}$	=	Some $\bar{q}$ is p	=	Not all $\bar{q}$ is $\bar{p}$
O'	Not all $\bar{p}$ is $\bar{q}$	=	Some $\bar{p}$ is q	=	Some q is $\bar{p}$	=	Not all q is p
I	Not all p is $\bar{q}$	=	Some p is q	=	Some q is p	=	Not all q is $\bar{p}$
I'	Not all $\bar{p}$ is q	=	Some $\bar{p}$ is $\bar{q}$	=	Some $\bar{q}$ is $\bar{p}$	=	Not all $\bar{q}$ is p

In this table column (i) gives propositions of the form *A* or *O* (which admit of simple contraposition); the second column is derived by obversion, giving propositions of the form *E* or *I* (which admit of simple conversion); the third is next derived by (simple) conversion, giving again propositions of the form *E* or *I*; and the fourth is derived again by obversion, giving propositions of the form *A* or *O*. The processes of obversion and simple conversion being reciprocal, give equipollents, i.e. formal co-implicants. The relation of contradiction (i.e. formal co-opposition) is seen in the first instance from the predesignations 'All' versus 'Not all' and 'Some' versus 'No,' and, on account of the equipollences tabulated, each of the four universal propositions in any the same row is related as contradictory to each of those in the ordinally corresponding row of particulars: i.e. *A* to *O*, *A′* to *O′*, *E* to *I*, *E′* to *I′*.

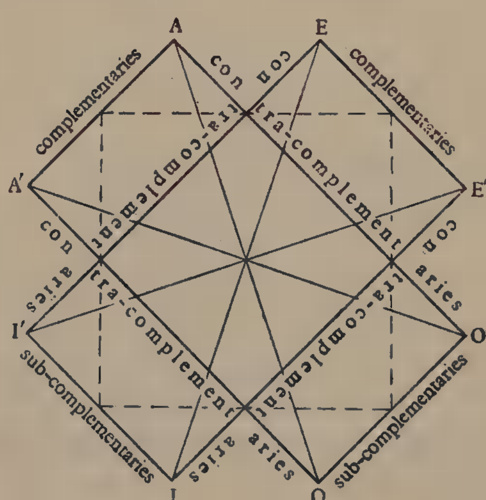
TABLE II

	A	*E*	*I*	*O*
Verse	All p is q	No p is q	Some p is q	Not all p is q
Obverse	No p is $\bar{q}$	All p is $\bar{q}$	Not all p is $\bar{q}$	Some p is $\bar{q}$
Contrapositive	No $\bar{q}$ is p	Some $\bar{q}$ is p		Some $\bar{q}$ is p
Obverted Contrapositive	All $\bar{q}$ is $\bar{p}$	Not all $\bar{q}$ is $\bar{p}$		Not all $\bar{q}$ is $\bar{p}$
Inverse	Some $\bar{p}$ is $\bar{q}$	Not all $\bar{p}$ is $\bar{q}$		
Obverted Inverse ...	Not all $\bar{p}$ is q	Some $\bar{p}$ is q		
Obverted Converse ...	Not all q is $\bar{p}$	All q is $\bar{p}$	Not all q is $\bar{p}$	
Converse	Some q is p	No q is p	Some q is p	
Verse	All p is q	No p is q	Some p is q	Not all p is q

Table II gives all the implications amongst the given propositions that can be drawn by alternate ob-

version and conversion, beginning first with obversion, and next with conversion. The arrows in the table show the direction of the inference, and where there is no arrow there is no inference. This table includes the equipollents of Table I; and contains also the sub-altern conversions which are required for the form A. When a proposition of the form O is to be converted the process must stop. The same table could be written out for A', E', I', O'. Having in this way given exhaustively the relations of equipollence, contradiction, sub- and super-implication and opposition, it remains to deal with the relation of *independence*. This will hold between the following pairs: A and A', E and E', A and O', A' and O, E and I', E' and I, for which we shall use the technical terms complementary for independence between universals, sub-complementary for independence between particulars, and contra-complementary for independence between a universal and a particular.

All these results are expressed in the following octagon of implication and opposition.



§ 6. The above account of the processes of sub-alternation, obversion and conversion of general categorical propositions is based upon the logical relations amongst compound propositions explained in Chapter III, by applying these latter without modification in the form of logical relations amongst adjectives. Now this correspondence may be further developed by bringing out the analogies between the universal and particular forms of categorical proposition on the one hand, and what was called in Part I Chapter III the necessary and possible forms of the compound proposition on the other hand. Thus, the form 'p implies q,' where the relation asserted of the two component propositions is irrespective of their truth or falsity, is naturally contradicted in the form 'p does not imply q,' where again the relation is asserted irrespectively of the truth or falsity of these components. Analogously, taking p and q to be adjectives (instead of propositions) the categorical 'Everything that is p is q,' where a relation of p to q is asserted irrespectively of any *given* thing being p or q, is naturally contradicted in the form 'Not everything that is p is q,' where again a relation is asserted of p to q irrespectively of any *given* thing being p or q. Expressing the compound propositions in terms of possible or impossible, the proposition 'p with not-q is impossible,' contradicts 'p with not-q is possible,' these compounds being respectively analogous to the universal 'Nothing that is p is non-q' and the particular 'Something that is p is non-q.' Thus there is a literal equivalence in the relations subsisting amongst the 'necessary composites' and 'possible conjunctives' on the one hand, and those subsisting amongst the 'universal' and 'particular' cate-

goricals on the other hand. Some logicians indeed have demanded that Logic should *interpret* the universal and particular categoricals to stand respectively for necessary implication and possible conjunction; this view, however cannot be accepted. The analogy that properly holds demands equivalence—not in the forms of proposition themselves—but in the logical relations amongst them. The basis for these analogies is shown in the following fundamental forms, where p and q are to stand for adjectives:

	General Categorical Form		Adjectivally Compound Form		
A.	Every p is q	=	p	implies	q
A'.	Every q is p	=	p	is implied by	q
E.	No p is q	=	p	is co-disjunct to	q
E'.	Everything is p or q	=	p	is co-alternate to	q
O.	Not every p is q	=	p	does not imply	q
O'.	Not every q is p	=	p	is not implied by	q
I.	Some p is q	=	p	is not co-disjunct to	q
I'.	Not everything is p or q	=	p	is not co-alternate to	q

The four relations and their contradictories here exhibited lead by combination to seven possible relations corresponding exactly to those shown in Part I Chapter III. Thus:

A and *A'*. Every p is q and Every q is p	= p	is co-implicant to	q
A and *O'*. Every p is q and Not every q is p	= p	is super-implicant to	q
A' and *O*. Every q is p and Not every p is q	= p	is sub-implicant to	q
O and *O'* and *I* and *I'*.	= p	is independent of	q
E' and *I*. Everything is p or q and Some p is q	= p	is sub-opponent to	q
E and *I'*. No p is q and Not everything is p or q	= p	is super-opponent to	q
E and *E'*. No p is q and Everything is p or q	= p	is co-opponent to	q

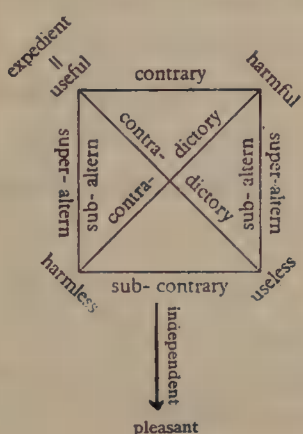
Thus the same seven relations in which propositions may stand to one another hold of the relations in which

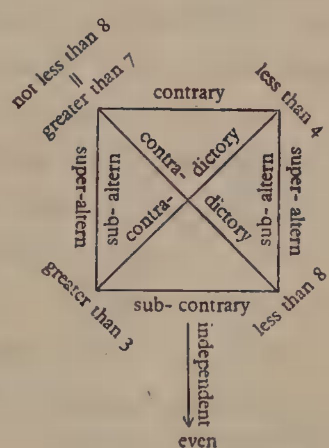
adjectives may stand when entering into the subject and predicate of universal and particular propositions.

If p is formally independent of q, then all the relations tabulated above are material, but if p is formally related to q, then it must be related in one or other of the six ways, which remain when the relation of independence is excluded. For example, taking the five adjectives *useful*, *harmful*, *useless*, *harmless*, and *expedient*, which are formally related, and adding *pleasant* which is formally unrelated, we find that:

(i) expedient is equipollent to useful,
(ii) useful is super-altern to harmless,
(iii) useless is sub-altern to harmful,
(iv) [useful is independent of pleasant],
(v) useless is sub-contrary to harmless,
(vi) useful is contrary to harmful,
(vii) useful is contradictory to useless.

The same relations hold for the five numerical adjectives—greater than 7, less than 4, less than 8, less than 3, not less than 8, to which we may add 'even' in place of 'pleasant.' Thus:





§ 7. Yet another way of representing the categorical proposition is in terms—not of the adjectives p and q—but of the substantive classes P and Q which these adjectives determine. Any class must be conceived in extension as a part of the universe of substantives, where the universe is sufficiently widely extended to include all the classes which occur in any set of interconnected propositions. The absolutely widest substantive universe is that which we have represented by the word 'thing'; and corresponding to any more restricted universe, the same word 'thing' can be used with a correspondingly understood restriction.

The following three technical terms may now be introduced: (*a*) the part of the universe which remains when any given class is subtracted will be denominated the *remainder* (or *co-remainder*) to the class; (*b*) the smallest class that *includes* both of two given classes P and Q will be denominated 'P with Q'; and (*c*) the largest class that *is included in* both of two given classes P and Q will be denominated 'P into Q[1].' With the help of these three class functions, the following fundamental relations between the class functions and the adjectival functions which determine them may be expressed:

(1) The class determined by the negative not-p = the remainder class to P.

(2) The class determined by the adjectival alternation 'p or q' = the class 'P with Q.'

(3) The class determined by the adjectival conjunction 'p and q' = the class 'P into Q.'

[1] These two functions of P and Q have many of the properties of the arithmetical L.C.M. and H.C.F. See also Chapter VIII.

Thus the notions 'not,' 'or,' 'and' which must be applied to predications (and here to adjectives) *correspond* respectively to the notions 'remainder,' 'with,' 'into,' which apply to substantive classes. Traditional logic has encouraged confusion between these two types of notion by employing the terms which are only proper for adjectival functions, for class functions also[1]. This usage, while it has the advantage of brevity and facilitates the logical transformations that the learner has to carry out, suffers from the serious objection of leading to confusion between the two types of function. Thus the notion of the remainder as a relation between classes is founded upon that of non-identity as a relation between substantive individuals. For when the class X is said to be the remainder to the class Y, part of what is meant is that no individual comprised in the one class is identical with any individual comprised in the other. Now this relation of non-identity has been repeatedly confounded with that of negation; so much so, that an important school of philosophy seems to hold that diversity or non-identity involves *prima facie* a contradiction; in other words, that the togetherness of non-identical substantives in the universe of reality involves the joint affirmation and negation of one and the same predicate.

The correspondences between adjectival relationships and class relationships will now be shown by first taking each of the four universals A, A', E, E'. Thus:

[1] The reversed confusion is committed when adjectival predications are spoken of as 'co-exclusive' or 'co-exhaustive' instead of 'co-disjunct' or 'co-alternate.'

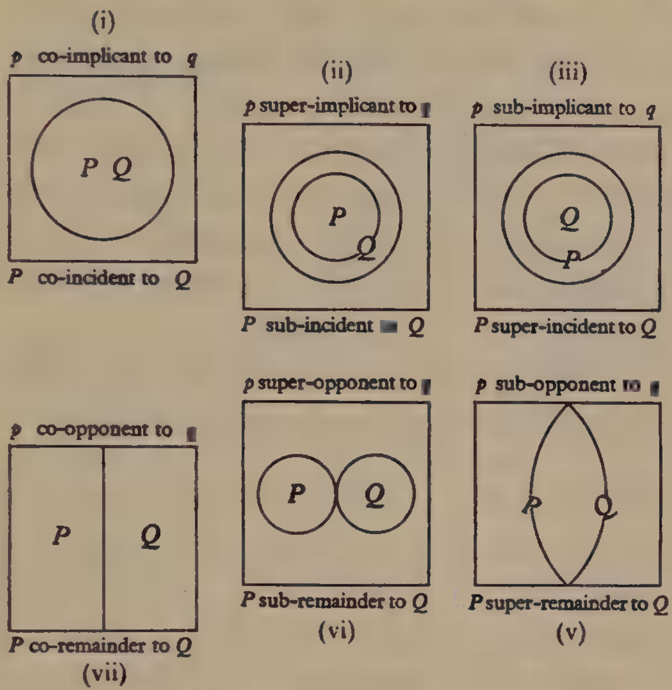
	Adjectival Relations	Distributive Relations	Class Relations
Direct Implicative	p implies q	*A*. Every p is q	P is-included-in Q
Counter Implicative	p is implied by q	*A′*. Every q is p	P includes Q
Disjunctive	p and q are co-disjunct	*E*. No p is q	P and Q are co-exclusive
Alternative	p and q are co-alternate	*E′*. Everything is p or q	P and Q are co-exhaustive

Since *O*, *O′*, *I*, *I′* respectively contradict *A*, *A′*, *E*, *E′*, we derive the following combinatory results:

(i) p is co-implicant to q	(*A* and *A′*.) Every p is q *and* Every q is p	P is co-incident to Q
(ii) p is super-implicant to q	(*A* and *O′*.) Every p is q *and* Not every q is p	P is sub-incident to Q
(iii) p is sub-implicant to q	(*A′* and *O*.) Every q is p *and* Not every p is q	P is super-incident to Q
(iv) p is independent of q	(*O* and *O′* and *I* and *I′*.) Some but not every p is q *and* Some but not every non-p is non-q	P is inter-sectant to Q
(v) p is sub-opponent to q	(*E′* and *I*.) Everything is either p or q *and* Something is both p and q	P is super-remainder to Q
(vi) p is super-opponent to q	(*E* and *I′*.) Nothing is both p and q *and* Not everything is p or q	P is sub-remainder to Q
(vii) p is co-opponent to q	(*E* and *E′*.) Nothing is both p and q *and* Everything is either p or q	P is co-remainder to Q

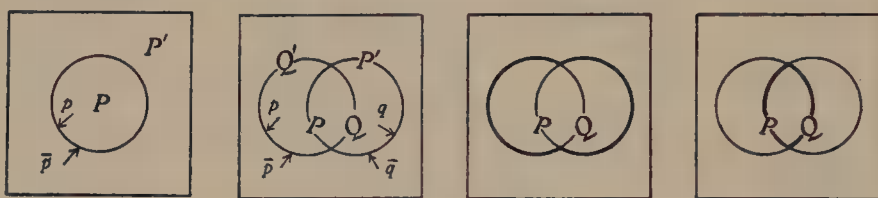
Here the class-relationships must be compared and contrasted with the adjective-relationships. In particular a 'super'-relation for the adjectives always yields a 'sub'-relation for the classes—illustrating the general principle that a *more determinate* connotation yields a narrower denotation, and a *less determinate* connotation

yields a wider denotation. The above seven relations may be at once expressed on Euler's scheme. Thus:



These diagrams, due to Euler, illustrate the adaptation of diagrammatic representation to propositions expressed, as above, in terms of classes. The employment of diagrams in Logic requires some special explanation. A class is represented by a closed figure, while anything that is comprised in the class may be represented by a point within this figure; and anything not comprised in the class, by a point outside the figure. It is further requisite that the all-inclusive class (otherwise called the universe) whether restricted or unrestricted should be represented also by a closed figure within which all the specific classes adjectivally delimited should fall. Thus the boundary line may be taken to represent the adjective by which the scope of the class is determined,

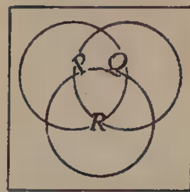
while the area within this boundary-line represents the class itself. In the figure below the class P is represented as determined by the adjective p; the universe is represented by the square, and what is outside the circle represents the class-remainder to P, which will be symbolized as P', the boundary of which is indicated by $\bar{p}$. For two adjectives p and q, we must use two areas having a part in common with a remainder to both. The thickened outline in (3) separates the class 'P with Q' from 'P' into Q''; and the thickened line in (4) separates the class 'P into Q' from 'P' with Q'.' The diagram shows to the eye the correspondences between



(1) (2) (3) (4)

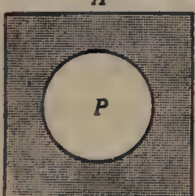
the adjectival and the class functions; viz., that the class determined by the adjectival *alternation* 'p or q' is the class 'P with Q,' and that determined by the adjectival *conjunction* 'p and q' is the class 'P into Q.' These diagrams, first employed by Dr Venn, do not represent any proposition, but the framework into which propositions may be fitted. Thus it is shown, for instance, that, using two determining adjectives—p and q—the universe is divided into 2×2 classes, namely: P into Q, P into Q', P' into Q, and P' into Q', determined respectively by the adjectival conjunctions 'p and q,' 'p and $\bar{q}$,' '$\bar{p}$ and q,' '$\bar{p}$ and $\bar{q}$.' Again: taking three determining adjectives p, q, r, we must draw three closed figures

in such a way that every resulting sub-class shall be represented; namely the $2 \times 2 \times 2$ classes P into Q into R, P into Q into R', P' into Q into R, P' into Q into R', P into Q' into R, P into Q' into R', P' into Q' into R, P' into Q' into R', as determined by the corresponding adjectival conjunctions.



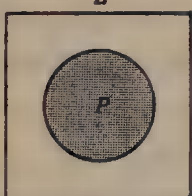
Into these frameworks propositions are fitted in the following manner. Beginning with a single determining adjective p, consider the four propositions: *A*. Everything is p; *E*. Nothing is p; *I*. Something is p; *O*. Not everything is p. These four propositions can be expressed in terms of the classes P and P' thus: *A*. P exhausts the universe or P' is empty; *E*. P is empty or P' exhausts the universe; *I*. P is occupied or P' does not exhaust the universe; *O*. P does not exhaust the universe or P' is occupied. The import of each of these propositions may therefore be expressed by means of the opposite conceptions of *occupied* and *empty*: crowded horizontal shading will be used to indicate empty, and a single straight line to indicate occupied:

A



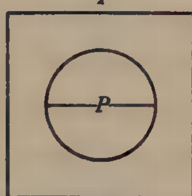
Everything is p

E



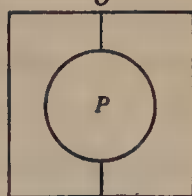
Nothing is p

I



Something is p

O

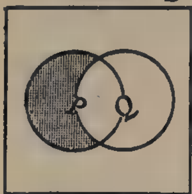


Not everything is p

This shows that the universals may be expressed as denying and the particulars as affirming occupation. Taking now two adjectives p and q, we have eight distinct propositions *A*, *A'*, *E*, *E'*, *O*, *O'*, *I*, *I'*, where

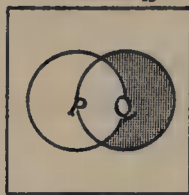
again the universals deny and the particulars affirm the occupation of certain sub-classes.

P is included in *Q*



A_n Every *p* is *q*

P includes *Q*

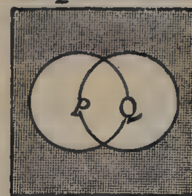


A'_n Every *q* is *p*

P and *Q* are co-exclusive

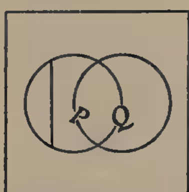
E_n No *p* is *q*

P and *Q* are co-exhaustive

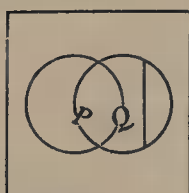


E'_n Everything is *p* or *q*

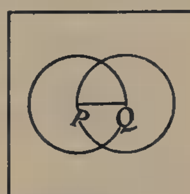
These are respectively contradicted by



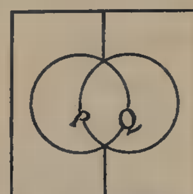
O_f Not all *p* is *q*



O'_f Not all *q* is *p*



I_f Something is *p* and *q*



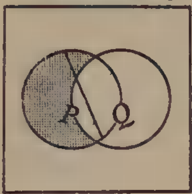
I'_f Not everything is *p* or *q*

In this system the strict relation of contradiction is indicated as of O to A, O' to A', I to E, I' to E', by the single straight line and the shading occurring in the same sub-class, namely PQ', $P'Q$, PQ, $P'Q'$.

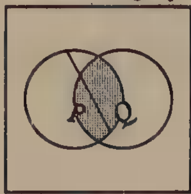
It must be observed that in Venn's system (1) the circles are drawn in every case as overlapping one another without exhausting the universe, and (2) that the specific proposition is represented by marking some sub-class as occupied or as empty.

Next let us combine the diagrams representing the several universals, which contain *no* instantial affirmation, with the diagram representing 'Something is p' for A and E, and 'Something is not-p' for A' and E'; or again with the diagram representing 'Something is $\bar{q}$' for A and E', and 'Something is q' for A' and E.

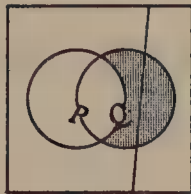
A Nothing is $p\bar{q}$
and something is p



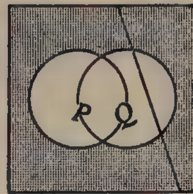
E Nothing is pq
and something is p



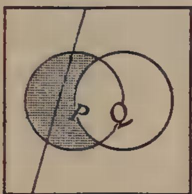
A′ Nothing [redacted] $\bar{p}q$
and something is $\bar{p}$



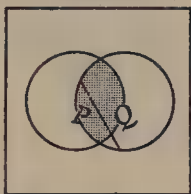
E′ Nothing is $\bar{p}\bar{q}$
and something is $\bar{p}$



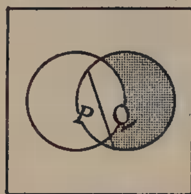
A Nothing is $p\bar{q}$
and something is $\bar{q}$



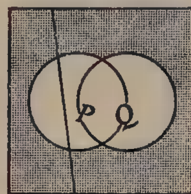
E Nothing is pq
[redacted] something is [redacted]



A′ Nothing is $\bar{p}q$
and something is q



E′ Nothing is $\bar{p}\bar{q}$
and something is $\bar{q}$



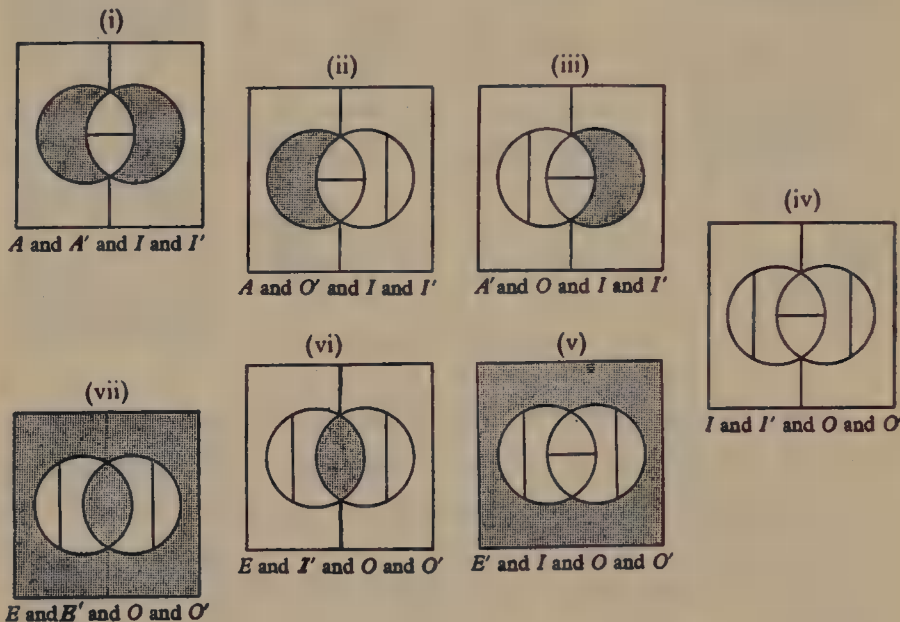
Now since a single line means that something is to be found in one *or* other of the two sub-classes which it crosses, and since the shading denies it for *one* of the two, it follows that something is to be found in the other. Thus the first four diagrams prove to the eye respectively that:

From 'Every p is q *and* Something is p' we can infer *I*. 'Some p is q'
From 'No p is q *and* Something is p' we can infer *O*. 'Some p is non-q'
From 'Every q is p *and* Something is non-p' we can infer *I′*. 'Some non-p is non-q'
From 'Everything is p or q *and* Something is non-p' we can infer *O′*. 'Some non-p is q'

And similarly for the second four, where the affirmative instantial involves non-q or q.

These inferences illustrate the general principle that in order to infer a proposition giving instantial affirmation, we must have a premiss giving instantial affirmation. From A_n *alone* we cannot infer I_f, but from A_n jointly with 'Something is p' we can infer I_f.

Finally let us use Venn's diagrams to represent the seven relations which result from the possible combinations of the eight elementary propositions A_n, A_n', E_n, E_n', I_f, I_f', O_f, O_f'.



In comparing this scheme with that of Euler (previously given) two points arise. Euler draws these figures from the outset so as to represent the class-relationships both as regards instantial affirmations and instantial denials, so that the figures directly express propositiona information. But in Venn both these factors in the proposition have to be specifically marked and in order to represent a completely determined class-relationship all the four sub-classes must be marked. In spite of this apparent difference, an optical comparison of this last scheme with the Eulerian scheme on p. 149 will disclose their essential equivalence. The practical distinction, however, remains that in Euler's scheme each uncom-

pounded categorical must be represented by an *alternative* of figures, viz.:

A by (i) or (ii); and O by (iii) or (iv) or (v) or (vi) or (vii)
A' by (i) or (iii); and O' by (ii) or (iv) or (v) or (vi) or (vii)
E by (vi) or (vii); and I by (i) or (ii) or (iii) or (iv) or (v)
E' by (v) or (vii); and I' by (i) or (ii) or (iii) or (iv) or (vi)

and conversely,

each diagram represents a *conjunction* of propositions

(i) $= A$ and A', (ii) $= A$ and O', (iii) $= A'$ and O,
(iv) $= I$ and I' and O and O',
(v) $= E'$ and I, (vi) $= E$ and I', (vii) $= E$ and E'.

On the other hand Venn's diagrams represent each of the uncompounded propositions by its appropriate 'marking' of the proper sub-class, and are thus immediately adapted to the conjunction of two or more affirmatively or negatively instantial pieces of information.

§ 8. All the above inferential operations are performed upon adjectival factors, these occurring always as predicates in a principal or subordinate clause; and, as is impressively brought out in the so-called 'existential' formulation of the proposition, a residual substantival factor always remains in the subject, though for linguistic convenience it may appear also in the predicate. The importance of this feature may have been obscured owing to the complicated detail with which the inferences have been treated; and, in conclusion, it is therefore to the point to emphasize the connection between the account of inference in this chapter and that of the functioning of substantive and adjective given in Chapter I.

CHAPTER X

EXISTENTIAL, SUBSISTENTIAL AND NARRATIVE PROPOSITIONS

§ 1. Before directly approaching the topic to be dealt with in this chapter, it will be necessary to consider the general question of the classification of propositions. In previous chapters, several classifications of propositions under different *fundamenta divisionis* have been given: for example they have been divided into simple and compound, the latter being subdivided into conjunctive and composite; and again into uncertified and certified, the latter being subdivided into formally and experientially certified; even the distinction between true and false yields an exclusive and exhaustive division of propositions. On the other hand, many well-known so-called classifications of propositions break the purely formal rules of logical division, in that the sub-classes are not mutually exclusive, and often can hardly be regarded as collectively exhaustive. The most notorious example of this is the classification of propositions upon which Kant based his enumeration of the categories, and which comprised such sub-classes as singular, particular, universal, affirmative, negative, hypothetical, categorical, assertoric and problematic. Regarded as a classification of propositions this involves a flagrant violation of the formal rules of division; for a categorical proposition may be singular or universal, negative or affirmative, problematic or assertoric, etc. What is of real logical value, and was indeed intended

by Kant, is a classification not of propositions, but of the several formal relations which may enter within the structure of a proposition more or less simple or complex. For example, 'If the American harvest is bad, the European prices of corn are high' is properly enough denominated hypothetical because the central or principal relation asserted is that of implication; but further analysis discloses the categorical nature of the two implicationally related clauses, and the universality of the statement as understood to refer to any or every year; and furthermore such relations as contemporaneity and causality may be taken as implicitly asserted in such a proposition. Of distinctions amongst the different forms of relation that may enter into a proposition we may select as one of the most important that between the relation of characterisation and the relation of implication, which, properly speaking, should take the place of the distinction amongst propositions between categorical and hypothetical. Of these two relations—characterisation and implication—the former holds only of an adjective to a substantive, the latter only of one proposition to another proposition. Again, the distinctions of modality are not, properly speaking, distinctions between propositions, but distinctions between the different adjectives that can be significantly predicated of propositions. In short the sole logical purport of a so-called classification of propositions is, by means of an *analysis* of propositions of various forms of complexity, to disclose the different modes in which their components are bound into a unity.

With special reference to the topic of this chapter we may pass to such logicians as Lotze, Bosanquet

and others who have attempted to classify propositions on philosophical rather than purely formal principles. In particular Sigwart has distinguished propositions under three—not necessarily exclusive or exhaustive—heads corresponding to what, in the title, we have called existential, subsistential and narrative.

§ 2. We proceed then to examine in the first place what is meant by an existential proposition. The most general and appropriate sense in which the word 'existential' is predicated of a proposition is where the proposition refers to existence in that narrower sense in which existence is distinguished from subsistence, as two sub-divisions of reality. Thus the proposition '3 plus 4 equals 7' must be regarded not as existential, but as subsistential, if that terminology be permitted. From an examination of the illustrations given by philosophers of the existent (as distinct from the subsistent) it may be gathered that the term is equivalent to that which is manifested in time or space. This interpretation may be justified by considering the etymology of the word 'existent' which is closely connected with 'external,' and is further confirmed by the fact that the typical so-called external relations are temporal or spatial. On the other hand it has been maintained, for example, that the number 3, or the relation of equality between 3 plus 4 and 7, or the relation of causality subsists rather than exists. If this conception is to be generalised what subsists is primarily an adjective, whether ordinary or relational; whereas what in the more exact sense may be said to exist is a substantive proper. We may therefore regard the terms 'existent' or 'substantive proper' as meaning 'what is manifested in time or

space.' Thus an existential proposition is distinguished from a subsistential proposition in that the latter makes predications about adjectives (including propositions) as such. It may, however, be maintained that such a proposition as '3 plus 4 equals 7' should properly be interpreted as existential on the ground that it applies to all possible existent groupings of classes numbered 3, 4, and 7; or again that the proposition 'Unpunctuality is irritating' is existential on the ground that it means nothing more nor less than that 'All unpunctual arrivals are irritating,' where the term arrival, with its implicit temporal and spatial reference, obviously stands for an existent. Or yet again that the proposition 'Heat causes wax to melt' is existential on the ground that it merely expresses the universal proposition that 'All cases in which heat enters wax, are cases in which the wax is melted,' where merely temporal and spatial relations of possible occurrences are involved. If then a subsistential proposition is to be distinguished from an existential, it must be on the ground that propositions in which the *explicit* predications concern adjectives or relations, have a special significance beyond what they undoubtedly imply existentially[1].

§ 3. But turning from the more philosophical to the strictly formal usage of the term existential, we find that by such logicians as Venn, Keynes and Russell, existential and subsistential propositions are indifferently denominated *existential*, and that the term existential is used without any reference to the *substance* of the proposition, but rather to a certain mode in which

[1] This question will be treated in more detail in a subsequent chapter.

any *general* proposition (particular or universal) may be *formulated.* This entirely distinct and peculiar use of the term 'existential' has given rise to endless confusion; and, on this account, the term should be entirely discarded and replaced by some such term as *instantial,* or, more accurately, indeterminately instantial. At this point we must explain the distinction between determinately instantial and indeterminately instantial. While the former corresponds roughly to narrative propositions, of which we shall treat later, the latter are most naturally prefixed by the phrase 'there is' or 'there are'; e.g. there is a God; there are horses; there are no sea-serpents; there is an integer between 3 and 5; there are prime integers between 4 and 15; there is no integer between 3 and 4. Of these, the first three would be called existential, in the philosophical sense, the last three subsistential. A minor distinction amongst such indeterminately instantial propositions, the disregard of which has not infrequently led to confusion, is that between the affirmatively instantial and the negatively instantial. In short, the essential nature of a particular or of a universal proposition is expressed by formulating the former as affirmatively instantial, and the latter as negatively instantial[1].

The further development of the topic of such propositions, or rather of such propositional formulations, requires us to introduce the phrase 'universe of discourse,' to which frequent reference is made in formal expositions of the so-called existential import of propositions. There are two applications of this phrase, which demand different criticisms. One quite harmless

[1] See Chapter VIII.

application of the expression 'universe of discourse,' points merely to the familiarly elliptical nature of conversation. Thus the reference of such a proposition as 'All voters are males' is understood to be limited, say, to the present time (1914), the English nation, and election to Parliament. In spite of the fact that some nations now and all nations will confer the franchise upon women, in spite of the fact that for the Board of Guardians and other offices women take part in voting, the proposition 'all voters are males' is perfectly intelligible in its context. The phrase, though elliptical—like all phrases in discussion or conversation—does not require the explicit introduction of every well-understood qualification. In our view, therefore, logicians have unnecessarily paraded this application of the notion of a universe of discourse, where it means merely that the ordinary reader is expected to supply the restrictions intended by the writer. A limited universe understood as indicating the subject-matter of a single work, such as geometry, which refers exclusively to spatial figures, illustrates the same simple relation to the universe as a whole. Understood in this general sense, the universe of discourse has to *the* universe the relation of part to whole, and the notion is certainly harmless if trivial.

But the other application of the phrase requires more serious criticism. Here 'the universe of discourse' is presented to the reader, not as inside, but as outside what is commonly called *the* universe. In this usage the phrase seems to imply that there are several universes related to one another as Europe is to Africa rather than as France is to Europe, taking Europe in both cases to stand for *the* universe. The distinction

between the two uses of the phrase is evident when we pass from such an example as 'All voters are males' to 'Some fairies are malevolent' which well illustrates the use now under consideration. The former is understood to refer to a limited part of the universe of persons, whereas the latter refers to no part whatever of this universe, and on this account is said to be concerned with the 'universe of imagination' conceived as outside and separate from the universe of reality. Now if, when speaking of specific universes, such as the universe of imagination, the universe of ideation, and the universe of physical reality, we meant merely universes comprising images, ideas, or physical realities, then all these three are in the strictest sense included within the one single universe of existents, to which they are related merely as parts to whole. Anything that is comprised in the universe of images must be an image; anything that is comprised in the universe of ideas must be an idea (both of these being psychical); and similarly, anything comprised in the universe of physical realities must be physical. What logicians seem to have confused, and requires only common sense to distinguish, is between a horse and either the idea of a horse or the image of a horse; and accordingly a proposition about horses is concerned with different material from any proposition about ideas or images of horses. When then a proposition is spoken of as being false in the universe of reality and yet true in the universe of imagination or ideation, this involves the tacit assertion that the *same* proposition can be both true and false; whereas in fact the contents of the two propositions, one of which is said to be true and the other false, are different. The affirmation

or denial that there are sea-serpents is different from the affirmation or denial that there are images of sea-serpents; which again is different from the affirmation or denial that there are ideas of sea-serpents. It is absurd to say that the same things exist in one universe and do not exist in another: wherever this appears to be the case the things asserted or denied to exist are different. What is here said of sea-serpents holds equally of horses or of dragons; as regards the latter, it is supposed that because dragons are acknowledged not to exist in the universe of physical reality, there must be some universe in which they do exist in order that we may intelligently use the term 'dragon.' Now it is a purely psychological question whether at this or that moment of time an image of a horse or equally of a dragon is in course of mental construction; it may be that we may intelligently read or think about dragons or horses without mentally constructing any images of such creatures. Properly speaking there is no such thing as *the* image of a horse or *the* image of a dragon, because the constructing of images by one person at one time is, as an occurrence, distinct from such a construction by another person at another time, however closely these images may agree with one another in character. Hence, if existence is predicated of any image of this or that kind, it must be remembered that by existence is here meant manifestation in time, and that therefore there exist as many images of any kind of thing as there are occurrences of the constructive act.

What holds of images holds, strictly speaking, also of ideas, though not so obviously; the existence or non-existence of the idea of any object, if idea stands

for mental process, must mean the occurrence or non-occurrence of an act of thinking about the object during this or that period of time. The term 'idea,' however, may be understood in a less literally psychological sense: thus intelligently to entertain a proposition in thought would seem to entail our entertaining ideas corresponding to the several terms in the proposition. But in this connection we may refer to Mill's pronouncement in regard to the import of propositions in relation to ideas. His dictum is that propositions are not about ideas, but about things; and by this he intended to assert that a proposition is concerned with the things which it expressly talks of, and not with any mental process that may be involved in the assent to or understanding of the proposition. In short, although any genuine act of assertion requires as a preliminary process the understanding of the terms and combination of terms that constitute a proposition, yet it is not this process to which the proposition refers. This, of course, holds, whether the matter of the proposition is physical reality or mental reality: we must understand what is meant by the association of ideas or by an emotion of anger in a psychological proposition, just as we must understand what is meant by dragons or horses in propositions describing such creatures; while, on the other hand, the propositions in neither case are concerned with these processes of understanding.

§ 4. We next proceed to consider in what sense truth and falsity can be predicated of propositions such as 'Some fairies are malevolent' or 'No Greek gods are without human frailties.' These may be otherwise rendered: 'There are malevolent fairies,' 'There are no

Greek gods without human frailties.' If these be taken literally, as merely primary propositions, nothing can be said but that the first is necessarily false and the second necessarily true, because there are actually no fairies and no Greek gods in the real universe. But within the real universe there are to be found *descriptions* of fairies and of Greek gods in stories or legends, and hence it may be *true* that some fairies have been *described* as malevolent, and it may be *false* that no Greek gods have been *described* as without human frailties. If then we distinguish these secondary propositions—to be recognised as such by the introduction of the word 'describe'—from the original primary propositions, the establishment of their truth or falsity is seen to depend upon special evidence. The universe of descriptions is simply part of the universe of reality; indeed it seems strangely to have escaped logicians that books and the persons who wrote them belong to one real world, and that therefore the universe to which we refer for verification of propositions concerning the descriptions of fairies or of Greek gods is simply and precisely the same universe as that to which we refer for verification of propositions concerning Frenchmen and geologists. It is therefore evident that, only when we have transformed such primary propositions into their secondary correspondents, any question of interest arises as to their truth or falsity. This contention finally forbids us to speak of various universes of discourse which are outside the one universe of reality. The briefest mode of indicating the peculiarity of propositions of the type illustrated is to say quite simply that they are elliptical; not elliptical in the sense of limiting the subject-term to a narrower sphere

included in the universe, but elliptical in the sense of being expressed as primary propositions and understood as secondary. Thus our first example should properly be expressed 'Story-books describe some fairies as being malevolent,' and our second 'Homer describes all the Greek gods as subject to human frailties'; and in these transformed shapes the propositions are seen at once to be verifiable in exactly the same way as any other propositions; namely by reference to the one real universe of books and persons.

§ 5. We pass now to the logical significance of the term narrative in its application to propositions. The notion of a narrative proposition is not restricted to the type of proposition characteristic of a work of fiction or history, since it includes statements made in ordinary conversation etc., where there may be no intention to develop the account of an incident into a connected story. Moreover histories and novels are composed of others besides narrative propositions—the non-narrative propositions being generally what we may call comments on the incidents, characters, situations or emotions described. Novels (or even histories) might indeed be classified according as their narrative or commentary elements predominate; compare for example Scott with Thackeray, or S. R. Gardiner with Macaulay. A narrative proposition may be more precisely defined as one whose subject-term is prefixed by introductory or referential applicatives; whereas non-narrative propositions are prefixed by such distributives as 'every,' 'some' or similar phrases. Now distributives serve as predesignations of adjectivally significant subjects, while commentary propositions may be distinguished from such

narrative propositions as may happen to use subjects containing an adjectival element because in the *latter* case the adjective has no general reference. For example, in the course of a narrative the proposition may occur 'A shabbily-dressed gentleman entered the room,' and this may be followed later on by 'the shabby gentleman withdrew,' where the adjective 'shabby' enters in the context merely first as introductory and later as referential. In fact it is not by the consideration merely of the grammatical structure of a sentence, but rather by the logical nexus of the propositions that the distinction can be established; the narrative and non-narrative elements in any literary work being not necessarily expressed in separable sentences. Hence the reader may easily pick out the commentary elements, these being recognisable by their reference to persons and things in general as distinct from the persons and things entering directly into the plot.

My account of narrative propositions covers a wider range than is apparently intended by Sigwart; but for both of us the distinction between narrative and non-narrative rests upon that between the substantive and the adjective. In Sigwart's definition the subject in the narrative proposition is merely substantival, while the subject as well as the predicate in the non-narrative proposition contains adjectival elements. My application of the term 'narrative' on the other hand, includes cases in which the subject term may contain an adjectival element the significance of which is purely introductory or referential.

§ 6. A new problem, bearing upon the existential import of propositions is raised when we contrast

fictitious with historical narratives. We may take for illustration the proposition 'Mr Pecksniff is a hypocrite' and first ask what is meant by Mr Pecksniff. Now a provisional answer to this would be 'A certain architect, living near Salisbury, in the beginning of the nineteenth century.' The term by which we have replaced Mr Pecksniff seems to have an obvious reference to the universe of reality, and more particularly to the universe of things happening and existing in time and place. But the question as to whether there was any architect in Salisbury at that time would be irrelevant, and therefore the proposition would appear not to be about any architect then living near Salisbury. The difficulty here points to a peculiarity in the use of the predesignation 'a certain.' If for Pecksniff we had substituted 'some architect' instead of 'a certain architect,' the proposition 'Some architect living then near Salisbury was a hypocrite' would have been amenable to the ordinary modes of verification. But the form of statement 'A certain architect was a hypocrite' appears not to represent a proposition, inasmuch as it cannot be either affirmed or denied, since the architect to whom the writer refers is not indicated. What holds then of the reader or hearer of such a proposition does not hold of the writer or speaker[1]. Though the hearer is unable to give the direct contradictory of the proposition, yet the speaker may propound the two alternatives

[1] According as logicians exclusively interpret propositions from the point of view of the speaker (writer) or hearer, they are to be classed respectively as conceptualists or nominalists. The difference between these two points of view lies at the root of many logical controversies.

that a certain S is or is not P, provided that he has his own individual means of identifying the S to whom he is referring in thought. In fact the most common usage of the phrase 'a certain' involves deliberate concealment for various harmless purposes on the part of the speaker. Thus, when I say 'A certain boy now in this room has stolen my purse,' I deliberately preclude any hearer from strictly contradicting or agreeing with me, though of course he could deny the proposition by asserting not the contradictory but a contrary, namely: 'No boy in this room has stolen the purse.'

As regards a narrative, fictitious or historical, however, where any substantival reference must always be interpreted in accordance with its nexus with the introductory 'a certain' (coupled or not with a proper name), the writer and reader are so far in the same position that neither the one nor the other is concerned with the question of ultimate identification. The referential 'the' is prefixed to an object identical with that to which the introductory 'a' was first prefixed, but outside and beyond this nexus there is no further possibility of identification. Hence the whole body of propositions in a fictitious narrative is not entertained with a view to the consideration of their truth or falsity, and might be called pure suppositions. The scholastic logicians introduced the phrase 'suppositio materialis' which would illustrate the sense in which 'supposition' has just been used; but modern logicians have interpreted this phrase as equivalent to what they call the universe of discourse.' It is obvious, however, that the two conceptions are totally distinct, inasmuch as the former consists of

classes of propositions included in the universe of all possible propositions, whereas the latter consists of classes of substantives included in the universe of all possible substantives. In contrasting a work of fiction with an historical work, the propositions laid down in the latter are put forward as to be accepted as true on the authority of the writer. But in both cases, whether history or fiction, it still holds that there is no means for ultimately identifying the characters introduced either on the part of the reader or the writer; we can only say that in history it is believed that these characters are identifiable with persons who have actually existed, whereas in fiction no such belief is involved.

§ 7. Within propositions which are fictitious, the distinctions between those which introduce beings to which there are, and to which there are not, similar beings in the world of reality, gives rise to a further problem[1]. Thus we may contrast the various statements about the architect Mr Pecksniff in *Martin Chuzzlewit* with the various statements that might be made about the fairy Puck in a fairy-tale. It will be noted that in the former case the general class (architect) to which reference is made, actually exists, whereas in the latter the class (fairy) to which reference is made, does not exist; while neither the individual Pecksniff nor the individual Puck does or ever did exist. This immediately gives rise to the question of the distinction or

[1] Of course there is a further distinction between fictions (novels, dramas) which describe characters and incidents such as *might* occur in the real world, and fairy-tales (myths, legends) which give descriptions such as never can occur, in that judgments as to their naturalness or 'realism' in the former case are more often relevant than in the latter.

relation in the significance of the word 'exist' as applied, firstly to a class, and secondly to an individual. We have above pointed out the peculiar and comparatively modern application of the word 'exist' on the part of formal logicians, who express the proposition: 'There are architects' in the form 'The class architect exists'; and the proposition 'There are no fairies' in the form 'The class fairy does not exist.' A more precise formulation of these propositions is obtained by taking C to stand illustratively for any class, the affirmation of whose existence is thus rendered: 'There is at least one individual, say P, which is comprised in C'; or rather, since a class is determined by connotation: 'There is at least one individual, say P, which is characterised by the conjunction of adjectives constituting the connotation of the name C.' Now here, I maintain, that the symbol P stands for a proper or uniquely descriptive name, and hence that the conception of the existence of a *class*—indicated by a connotative name—requires the conception of the existence of an *individual*—indicated by a proper or uniquely descriptive name. Now we may agree that there is no such individual as Pecksniff, and that there is no such individual as Puck; although in the first case the *class* 'architect'—which might be used in the description of Pecksniff—would be said to exist, while the class 'fairy'—which might be used in the description of Puck—would be said not to exist. If then we brought forward Sir Christopher Wren and Mr Pecksniff as instances of architects, or Oliver Cromwell and Mr Pecksniff as instances of hypocrites, would this substantiate the affirmation that there are at least *two* individuals comprised in the class

architect, or in the class hypocrite? If this question is answered in the negative it must be on the ground that, in some sense of the term 'exist' which is not appropriate to *classes*, Mr Pecksniff does not and never did exist, and hence he cannot count as one when we are enumerating the members comprised in any given class. Furthermore, since the numerical predication 'at least one' is highly indeterminate and could be in this or that case replaced by the relatively determinate 'at least n' where n stands for this or that number, the affirmation that 'the class C exists' is only a special and less determinate case of the affirmation that 'the class C comprises at least n items,' and the number n cannot be counted as such unless all the n items *exist*. The conclusion therefore follows that the sense of the word 'exist' when predicated of a class is dependent upon that of the word 'exist' when predicated of an item or individual indicated by a proper or uniquely descriptive name[1].

[1] This contention is directed against the position held in the *Principia Mathematica*, where E! is ultimately defined in terms of $\exists$, whereas in my view $\exists$ is to be ultimately defined in terms of E!

CHAPTER XI

THE DETERMINABLE

§ 1. In this chapter we propose to discuss a certain characteristic of the adjective as such, which perhaps throws the strongest light upon the antithesis between it and the substantive. Here it will be apposite to consider the traditional account of the principles of logical division where a class (of substantives) is represented as consisting of sub-classes. This process is governed by the following rules: (1) the sub-classes must be mutually exclusive; (2) they must be collectively exhaustive of the class to be divided; (3) division of the class into its co-ordinate sub-classes must be based upon some one 'fundamentum divisionis.' The first two of these rules may be said to be purely formal, and do not raise any problem of immediate interest; but the technical term fundamentum divisionis—though perhaps readily understood by the learner—is actually introduced without explicit account of its connection with, or its bearing upon, ideas which have entered into the previous logical exposition. To illustrate the notion we are told, for instance, that, when a class of things is to be divided according to colour, or to size, or to some other aspect in which they can be compared, then the colour, size, or other aspect constitutes the fundamentum divisionis. Now although, grammatically speaking, words like colour and size are substantival, they are in

fact abstract names which stand for adjectives; so that the fundamentum divisionis is, in the first place, an adjective, and in the second, an adjective of the particular kind illustrated by 'colour' when considered in its relation to red, blue, green, etc. Superficially this relation appears to be the same as that of a single object to some class of which it is a member: thus two such propositions as 'Red is a colour' and 'Plato is a man' appear to be identical in form; in both, the subject appears as definite and singular, and, in both, the notion of a class to which these singular subjects are referred appears to be involved. Our immediate purpose is to admit the analogy, but to emphasise the differences between these two kinds of propositions, in which common logic would have said we refer a certain object to a class.

I propose to call such terms as colour and shape *determinables* in relation to such terms as red and circular which will be called *determinates*; and, in introducing this new terminology, to examine the distinction between the relation of *red* to *colour* and the relation of *Plato* to *man*. To predicate *colour* or *shape* of an object obviously characterises it less determinately than to predicate of it *red* or *circular*; hence the former adjectives may be said negatively to be indeterminate compared with the latter. But, to supplement this negative account of the determinable, we may point out that any one determinable such as colour is distinctly other than such a determinable as shape or tone; i.e. colour is not adequately described as indeterminate, since it is, metaphorically speaking, that from which the specific determinates, red, yellow, green, etc., emanate; while

from shape emanate another completely different series of determinates such as triangular, square, octagonal, etc. Thus our idea of this or that determinable has a distinctly positive content which would be quite inadequately represented by the word 'indeterminate.' Further, what have been assumed to be determinables—e.g. colour, pitch, etc.—are ultimately *different*, in the important sense that they cannot be subsumed under some one higher determinable, with the result that they are incomparable with one another; while it is the essential nature of determinates under any one determinable to be comparable with one another. The familiar phrase 'incomparable' is thus synonymous with 'belonging to different determinables,' and 'comparable' with 'belonging to the same determinable'; not that this is the actual meaning of the terms, but that enquiry into the reason for the comparability or incomparability of two qualities will elicit the fact that they belong to the same or to different determinables respectively. This phrase 'belonging to' is also more usually used of a member of a class in relation to its class: we have, then, to contrast the significance of the relation 'belonging to' when applied in one case to a determinate and its determinable, and in the other to an individual and its class. If it is asked why a number of different individuals are said to belong to the same class, the answer is that all these different individuals are characterised by some the same adjective or combination of adjectives. But can the same reason be given for grouping red, yellow and green (say) in one class under the name colour? What is most prominently notable about red, green and yellow is that they are different, and even, as we may say, opponent

to one another; is there any (secondary) adjective which analysis would reveal as characterising all these different (primary) adjectives? In my view there is no such (secondary) adjective; in fact, the several colours are put into the same group and given the same name colour, *not* on the ground of any partial agreement, but on the ground of the special kind of difference which distinguishes one colour from another; whereas no such difference exists between a colour and a shape. Thus red and circular are adjectives between which there is no relation except that of non-identity or otherness; whereas red and blue, besides being related as non-identical, have a relation which can be properly called a relation of difference, where difference means more than mere otherness. What is here true of colour is true of shape, pitch, feeling-tone, pressure, and so on: the ground for grouping determinates under one and the same determinable is not any partial agreement between them that could be revealed by analysis, but the unique and peculiar kind of difference that subsists between the several determinates under the same determinable, and which does not subsist between any one of them and an adjective under some other determinable. If this is granted, the relations asserted in the two propositions 'Red is a colour' and 'Plato is a man,' though *formally* equivalent, must yet be contrasted on the ground that the latter but not the former is based upon an adjectival predication. For the latter is equivalent to predicating the adjective 'human' of 'Plato,' while, without denying that some adjectives may properly be predicated of (the adjective) red, yet the proposition 'Red is a colour' is *not* equivalent to predicating any adjective of red.

§ 2. Bearing in mind this distinction, the question arises whether what are called abstract names can be divided in the same way as concrete names into singular and general; in other words, whether adjectives can be divided into these two classes. The answer seems to be that adjectives can be divided into two classes more or less analogous to the singular and general which distinguish substantives, but that the two different kinds of adjectives are preferably distinguished as determinate and indeterminate. When, in considering different degrees of determinateness, the predication of one adjective is found to imply another, but not conversely, then the former we shall call a super-determinate of the latter and the latter a sub-determinate of the former. Thus the relation of super-determinate to sub-determinate means not only that the former is more determinate than the latter, but also that the predication of the former would imply that of the latter. A simple example can be taken from the determinable 'number': thus 7 is super-determinate to 'greater than 3'; the adjective 'greater than 3,' though not itself a *summum* determinable, may be called determinable, inasmuch as it is not merely indeterminate but capable of being further determined in the sense that it generates a definite series of determinates. To illustrate more precisely what is meant by 'generates'; let us take the determinable 'less than 4'; then 'less than 4' generates '3' and '2' and '1' in the sense that the understanding of the meaning of the former carries with it the notion of the latter. Now no substantive class-name generates its members in this way; take, for instance, 'the apostles of Jesus,' the understanding of this class-name carries with it the notion

'men summoned by Jesus to follow him,' but it does not generate 'Peter and John and James and Matthew etc.,' and this fact constitutes one important difference between the relation of sub-determinate to super-determinate adjectives and that of general to singular substantives.

§ 3. Another equally significant difference is brought out by considering that aspect of substantive-classes in which—to use the terminology of formal logic—increase of intension is accompanied by decrease of extension. The phrase 'increase of intension' conjures up the notion of adding on one attribute after another, by the logical process called conjunction; so that, taking p, q, r, to be three adjectives, increase in intension would be illustrated by regarding p, q, r conjoined as giving a greater intension than p, q; and p, q as giving greater intension than p. We have now to point out that the increased determination of adjectival predication which leads to a narrowing of extension may consist—not in a process of conjunction of separate adjectives—but in the process of passing from a comparatively indeterminate adjective to a comparatively more determinate adjective under the same determinable. Thus there is a genuine difference between that process of increased determination which conjunctivally introduces foreign adjectives, and that other process by which without increasing, so to speak, the number of adjectives, we define them more determinately.

In fact, the foreign adjective which appears to be added on in the conjunctive process, is really not introduced from outside, but is itself a determinate under another determinable, present from the start, though

suppressed in the explicit connotation of the genus. We propose to use a capital letter to stand for a determinable, and the corresponding small letter with various dashes to stand for its determinates. Thus, in passing from the genus p to the species pq, we are really passing from pQ to pq; or again the apparent increase of intension from p to pq to pqr is more correctly symbolised as a passing from pQR to pqR to pqr. In the successive process of dividing a summum genus into the next subordinate sub-genera, and this again into sub-sub-genera, the summum genus ought to be represented by a conjunction of determinables, say $PQRST$; the genera next subordinate to this, by $pQRST$, $p'QRST$, $p''QRST$, etc., and the genera next subordinate to the first of these by $pqRST$, $pq'RST$, $pq''RST$, and so on down to the *infima species* represented by determinates. Thus:

$$PQRST$$

$$pQRST \quad p'QRST \quad p''QRST\ldots\ldots$$

$$pqRST \quad pq'RST \quad pq''RST\ldots\ldots$$

In this way we represent from the outset the nature of the ultimate individuals under the summum genus, as being characterisable jointly by the determinables $PQRST$, while any genus or species is represented by these same determinables, one or more of which are replaced by determinates. This meets a criticism which has often been directed against the formal account of the inverse variation of extension and intension, since we see now that the same *number* of adjectives should be used in giving the connotation of the wider as of the narrower class. To illustrate these symbols from a botanical classification of plants: let the determinable

P stand for the number of cotyledons, Q for the disposition of the stamens, R for the form of the corolla, S for the attachment of the petals and sepals, and T for the divisibility of the calyx. Then $PQRST$ represents the summum genus '*plants*' as describable under these five heads, but otherwise undetermined in character. Then p, p', p'', might stand respectively for having no cotyledons, having 1, and having 2, thus representing the defining characteristic of each of the three classes—acotyledon, monocotyledon, and dicotyledon—by the symbols $pQRST, p'QRST, p''QRST$. Again q, q', q'', might stand respectively for the stamens being under, around or upon the carpels, thus representing the three sub-divisions—hypogynous, perigynous, epigynous—of dicotyledons, as $p''qRST, p''q'RST, p''q''RST$. Taking regular and irregular to be the two possible forms of corolla, then the next sub-division under $p''q'RST$ will be $p''q'rST$ and $p''q'r'ST$. Again s and s' may stand respectively for separability and inseparability of the calyx and corolla, and yield the further sub-divisions, say $p''q'rsT, p''q'rs'T$. The calyx may contain only one part or 3 or 4 or 5 or 6, and if these are represented respectively by t, t', t'', t''', t'''', a relatively determinate characterisation is finally symbolised by $p''q'rs't'''$ say.

There are cases for which a modification of this general scheme is required. The cases are those in which one particular sub-division is definable by the *absence* of an element upon which the predication of other determinables depend, while in the sub-divisions co-ordinate with this the element in question is present. For example, the class of plants called acotyledons might be defined by the absence of any corolla, etc., and

hence such variations as that of the form of the corolla or the disposition of the stamens, etc., are inapplicable to this particular sub-division.

§ 4. Now adjectives under the same determinable are related to one another in various ways. One relational characteristic holds in all cases; namely that, if any determinate adjective characterises a given substantive, then it is impossible that any other determinate under the same determinable should characterise the same substantive: e.g. the proposition that 'this surface is red' is incompatible with the proposition 'this (same) surface is blue.' It has been usual to modify the above statement by adding the qualification—at the same time and at the same place; this qualification applies where the substantive extends through some period of time and over some region of space, in which case the existent substantive, having temporal or spatial parts, may be said to be extended. For this reason the qualification would perhaps better be attached to the substantive itself, and we should say that, where opponent adjectives are predicated, reference is made to different substantives, since any one part of an extended substantive is existentially *other* than any other part.

A second characteristic of many determinates under the same determinable is that the differences between different pairs of determinates can be compared with one another; so that if a, b, c, are three determinates, there are cases in which we may say that the difference between a and c is greater than that between a and b; e.g. the difference between red and yellow is greater than that between red and orange. In this case the several determinates are to be conceived as

necessarily assuming a certain serial order, which develops from the idea of what may be called 'adjectival betweenness.' The term 'between' is used here in a familiar metaphorical sense derived from spatial relations, and is figuratively imaged most naturally in spatial form. Thus if *b* is qualitatively between *a* and *c*, and *c* qualitatively between *b* and *d*, and so on, the whole series has its order directly determined by the nature of the adjectives themselves. The further distinctions amongst series as interminable or as cyclic, and again of series of more than one order of dimensions, lead to logical complexities which need not be entered into here. Suffice it to say that this characteristic, which holds of so many determinates, gives significance to another well-known rule for logical division: *divisio non faciat saltum*: one meaning of which appears to be that we contemplate not merely enumerating a set of coordinate sub-classes, but enumerating them in a certain order. The rule prescribes that the order in which the sub-classes are enumerated should correspond to the order of 'betweenness' predicable of their differentiating characteristics.

The order of betweenness which characterises the determinates just considered may be either discrete or continuous. In the case of discrete series there is one determinate that can be assigned as next after any given determinate; but, in the case of a continuous series, a determinate can always be conceived as between any two given determinates, so that there are no two determinates which can be said to be next to one another in the serial order. It follows from this account of continuity that, between any two determinates which may be said to have a finite adjectival difference, may be interpolated

an indefinite number of determinates having a finite difference, and this number becomes infinite as the differences become infinitesimal. Amongst continuous series further differences between the interminable and the cyclic, and again between those of one or more order of dimensions, hold as in discrete series.

The reference here to determinables of higher or lower dimension requires explanation. Our familiar example of colour will explain the point: a colour may vary according to its hue, brightness and saturation; so that the precise determination of a colour requires us to define three variables which are more or less independent of one another in their capacity of co-variation; but in one important sense they are not independent of one another, since they could not be manifested in separation. The determinable colour is therefore *single*, though complex, in the sense that the several constituent characters upon whose variations its variability depends are inseparable.

§ 5. Returning to the conception of the absolutely determinate adjective, we have to note an important distinction between absolutely determinate and comparatively indeterminate predications. The distinction may thus be formulated: If, of two substantives the same *determinate* adjective can be predicated, then *all* the adjectives and relations definable in terms of the determinable, that can be predicated of the one, could be predicated of the other. But if, of two substantives the same *indeterminate* adjective can be predicated, then only *certain* of the adjectives and relations definable in terms of the determinable, that can be predicated of the one, can be predicated of the other. To illustrate

first the case of an indeterminate predication; let us take the numerical adjective 'greater than 7'; then of any collection of which this numerical adjective could be predicated, other adjectives such as 'greater than 5' and 'greater than 3' could also be predicated; but some collections that are 'greater than 7' such as the apostles, are greater than 11 and divisible by 4 for instance, whereas other collections that are 'greater than 7,' such as the muses, are less than 11 and are not divisible by 4: hence it is only *some* of the numerical adjectives that are predicable of the muses that are also predicable of the apostles, although the adjective 'greater than 7' is predicable of them both. Turning now to the case of *determinate* predication; if, instead of defining a collection by the indeterminate adjective 'greater than 7,' we had defined it by the determinate adjective 'twelve,' then any numerical adjective that is predicable of one collection of twelve, say the apostles, would be predicable of any other collection of twelve, say the months of the year or the sons of Israel; for example, 'greater than 11,' 'divisible by 4,' 'a factor of 96.' What we have here seen to hold of determinate and indeterminate number holds of any other determinable. The case of colour lends itself easily for illustration on account of the specific names which have been assigned to its determinates: thus, if the colours of two different objects are the same shade of yellow, then though these two objects may differ in any number of other respects such as shape and size, yet we may say that any colour-property of the one object will agree with the colour-property of the other; if the colour of one is more brilliant or less saturated than the colour of an orange,

then the same will hold for the colour of the other. In fact, whatever sensational determinable we take, whether it be colour, or sound, or smell, the determinate characterisations under any such determinable would lead to the same forms of generalisation that have been developed by science only in the sphere of quantity. It is agreed that in the sphere of sense perception, differences of quality strictly speaking hold only of the mental or sensational, and that the physical can only be defined in quantitative terms. Thus in the Weber-Fechner experiments the experient judges of equivalence or difference in the intensity or quality of his sensations, with which are correlated quantitative differences in the stimuli. The attempts that psychologists have made to discover formulae of correlation between the stimuli on the one hand and the sensations on the other hand show that determinateness in a qualitative or intensive scale carries with it the same logical consequences as does determinateness of magnitude for physically measurable quantities. Furthermore determinateness in either case is only approximately attainable, whether we rely upon the immediate judgments of perception or are able to utilize instruments of measurement. The practical impossibility of literally determinate characterisation must be contrasted with the universally adopted postulate that the characters of things which we can only characterise more or less indeterminately, are, in actual fact, absolutely determinate[1].

[1] The notion of the Determinable will be shown in later chapters to have importance in a large number of applications.

CHAPTER XII

THE RELATION OF IDENTITY

§ 1. The occasion for using the relation of identity is where a common term appears in different connections; thus we use the idea of identity always along with the idea of difference. The logical relation between difference—or more properly otherness—and identity, is that of co-opponency: that is, taking A and B as any two terms, it cannot be that A is both identical with and other than B, and it must be that A is either identical with or other than B. Thus the relation between identity and otherness is reciprocal. It must therefore be explained that we cannot *define* otherness as meaning non-identity any more than we can *define* identity as meaning non-otherness. The conceptions of identity and of otherness must be separately and independently understood before we can assert the above axioms.

The most trivial and apparently insignificant use of the relation of identity is expressed in the formula 'x is-identical-with x,' where what is primarily meant is that in repeated occurrences of the word x, either in a special context or irrespective of context, the word shall mean in any later occurrence what it meant in an earlier occurrence. Thus, even in this very elementary usage, the idea of identity goes along with the idea of otherness; for identity applies to what is *meant* by

the word, and otherness to its several occurrences. Underlying this characteristic of language there is the corresponding characteristic of thought; thus, in using 'x is-identical-with x' in reference to entities and not mere words, we return in thought to the object previously thought of, so that identity applies to what constitutes the *object* of thinking, and otherness to the several recurrent *acts* of thought.

A less elementary usage of the relation of identity occurs in the definition of words or phrases; thus, if x and y stand for two different phrases, we may speak of x being identical with y, although the phrases are palpably different. Here otherness applies to the phrases and identity to what is meant by the phrases. Verbal identification of x with y may be contrasted with factual identification; here the relation of identity applies (as before) to the objects denoted by the words; but the *proposition* asserting identity in the one case is of a different nature from the *proposition* asserting identity in the other case: for in the first case it is verbal, in the second factual. The relation of identity asserted in the two propositions: 'Courage is the mean between timidity and foolhardiness' and 'Courage is the one virtue required of a soldier' is the same: but the natures of the propositions differ, since the first—being put forward as a definition—is verbal, and the second is factual. More generally, we may distinguish the different grounds—such as rational, experiential or linguistic—upon which the assertion of any specific logical relation is based; but these differences in the grounds of assertion, do not affect the nature of the relation asserted. Thus, abbreviating 'is identical with'

into '*i*,' from the verbal statement *xiy* together with the factual statement *yiz* we may correctly infer the factual statement *xiz*. This inference uses the transitive[1] property of identity, and would therefore be impossible unless the relation of identity asserted in a verbal statement was the same as that asserted in a factual statement.

§ 2. We have shown the proper sense in which difference can be said to be involved in identity, but many philosophers have laid down the dictum that identity implies difference, in the sense, apparently, that when we assert that *A* is identical with *B* we are also involved in the assertion that *A* is different from *B*. The plausibility of this dictum depends upon a certain looseness in the application of the word 'implies'; thus the statement that identity implies difference is correct in the sense that the *asserting* of identity between one pair of terms implies our having implicitly or tacitly *asserted* difference between another pair of terms. This follows from what has been said above; e.g. when identifying the colour of this with the colour of that, we are implicitly differentiating 'this' from 'that'; and thus the identification and the differentiation may properly be said to be component parts of a single mental act. But, to give a more precise statement of this implication, it would be necessary to say that, when *A* is identical with *B in a certain respect*, then *A* is different from *B in some other respect*. In common language it is of course perfectly legitimate to say of two things that they are identical in respect

[1] For the term 'transitive' as a property of the relation of identity see Part I, Chapter XIV.

of colour and different in respect of shape. But here the term 'identity' should be used more precisely: we ought properly to say that, while the colour of this is identical with the colour of that, the shape of this is different from the shape of that. When then identity and difference go together in this way we ought to say, not that the *things* are both identical and different, but that one of their qualities is identical and another different. To complete the account of the modes in which identity and difference mutually involve one another without confusion, we need only take four such typical elementary propositions as:

(i) this is p,

(ii) this is q,

(iii) that is p,

(iv) that is q,

where p and q stand for different qualities, and 'this' and 'that' for different things. Then, comparing (i) with (ii) or (iii) with (iv), we have identity of thing and otherness of quality; and, comparing (i) with (iii) or (ii) with (iv), we have otherness of thing and identity of quality; finally, comparing (i) with (iv) or (ii) with (iii), we have both otherness of thing and otherness of quality.

§ 3. The relation and distinction between thing and quality may be generalised in terms of the correlative notions of substantive and adjective, the latter admitting of further resolution into determinates and their determinables. Thus, when predicating the same adjective p of this and of that substantive, we shall say that this and that *agree* as regards the determinable

P; and when predicating p of this and p' of that substantive, we shall say that this and that *disagree* as regards the determinable P. Here the term 'disagree' is used in place of 'differ,' for, strictly speaking, 'difference' is a relation between adjectives under the same determinable; and in measuring different degrees of difference amongst such adjectives, we may speak of the substantives as being *similar* when the degree of difference between the adjectives characterising them is small, and as being *dissimilar* when the degree of difference is great. Further we may say that two substantives partially disagree when they are characterised by the *same* determinates under certain determinables and by *different* determinates under certain other determinables. But partial agreement must be distinguished from approximate agreement, otherwise called similarity; and partial disagreement must be distinguished from remote disagreement, otherwise called dissimilarity. The distinction between similarity and dissimilarity involves reference to adjectives under the *same* determinable, and is obviously a matter of degree; while partial agreement or disagreement involves reference to adjectives under different determinables.

§ 4. Adjectives under the same determinable are usually said to be comparable; whereas those under different determinables are said to be incomparable or disparate. This point raises a question of considerable psychological interest as to the possibility of comparing two such disparate characters (say) as red and a trumpet-blast. The possibility of such comparison may perhaps be accounted for by association; or it may be that some

real deep-rooted form of connection underlies the two characters, which, if it could be explicitly rendered, would have its logical place amongst relations such as those which we are now discussing. But apart from this possible topic of psychological interest, it is not usual to speak of 'red' and 'a trumpet-blast' as being either like or unlike; and it is more usual to restrict the use of the terms *like* and *unlike* to qualities belonging to the same determinable, such as colour or sound. If this is admitted, the conclusion at once follows that like and unlike are not proper logical contradictories, but are *relations of degree*; so that to say of two comparable qualities that they are more or less like is equivalent to saying that they are less or more unlike; we cannot define the point at which the relation of difference changes from likeness to unlikeness, their opposition being only one of degree. When we compare different degrees of difference between determinates under a determinable whose variations are *continuous*, and judge, for instance, that the difference between *A* and *C* is greater than that between *A* and *B*, such differences between the determinates may be said to have *distensive magnitude*. When this distensive magnitude is too small, we fail perceptually to discriminate (say) between *A* and *B*; and some psychologists have virtually taken 'identity' to be equivalent, in such cases, to *minimum discernible difference*. This, however, entails logical contradiction; for the conception of a 'minimum discernible difference' implies that we fail to discriminate between qualities, which really *are* different and not identical; and that strict identity can only be predicated when difference has reached the

absolute limit zero. On this ground, it might be urged that identity when applied to qualities susceptible of continuous variation *means* zero difference; and has, therefore, a different significance from identity when applied to things or to qualities which vary discretely. This contention would, in my view, be fallacious; for it would appear to involve a confusion between the objective conception of identity itself and the subjective limitations in our power of judging identity. On the other hand, *difference* when applied to adjectives under the *same* determinable has a certain meaning which is distinct from any meaning of difference applicable to substantives or to adjectives under one and another determinable. As regards the latter, difference can only mean mere otherness; but as regards the former, difference may mean more than mere otherness; viz. something that can be measured as greater or smaller. Thus Socrates is merely other than Plato, red is merely other than hard; but round and square, red and yellow, five and nine are not merely non-identical, but are also such that the difference between them can be apprehended as greater or smaller (say) than that between oblong and square, orange and yellow, seven and nine.

§ 5. There is yet another aspect of the dictum: identity (of adjective) implies difference (of substantive) according to which it could equally well be rendered: difference (of adjective) implies difference (of substantive). For, where identity applies to the adjective and difference to the substantive, identity may properly be said to imply difference, in the sense that the identity predicated of an adjective is used along

with otherness as predicated of compared substantives; but, in this sense, we may also say that difference implies difference; i.e. that difference predicated between two adjectives is used along with otherness as predicated of the compared substantives. The dictum should therefore be expressed in more general terms to include identity and difference in respect of the adjectives characterising one and another substantive. Thus: Comparison with respect to any determinable character, whether it yields identity or difference, presupposes otherness of the substantives characterised by the determinable in question. In this connection we may examine the contrast commonly drawn between qualitative difference and numerical difference. This terminology is incorrect, for 'numerical difference' simply means otherness—the very notion 'numerical' owing its origin to the conception of mere otherness, which is the basis of number. Again in contrasting qualitative with numerical difference there is the suggestion that otherness does not apply to qualities or adjectives, whereas in its developments into number otherness is clearly seen to apply precisely in the same way to adjectives as to substantives. In our view the required distinction is that which was drawn above between the word difference as meaning merely otherness, and the word difference in its exclusive application to adjectives under the same determinable.

§ 6. Under the head of difference and otherness a special problem to be discussed is that involved in the famous Leibnizian principle 'the identity of indiscernibles.' This phrase signifies that 'indiscernibility implies identity,' which is an awkward way of saying that

'otherness implies discernibility.' Here the term *discernibility* has not a psychological but a purely ontological significance. More explicitly, the phrase signifies that a plurality of existent objects is only possible so far as there is some difference in the qualities or relations which can be predicated of them. If, in this phrase, the term relation is interpreted to include such external relations as space or time, then no reasonable criticism of the Leibnizian formula could be maintained; for, as has been contended from another point of view in Chapter II, 'existential otherness' implies difference in spatial or temporal relations. But this interpretation can hardly be taken to represent Leibniz's meaning, since he denied external relations, and held that this denial demonstrates the non-reality of space. But whereas he pretends to base the denial of space upon his dictum, in reality his dictum would have no plausibility unless it had been previously agreed that space was unreal. What Leibniz certainly meant was that *two* existent objects could not agree in all their *internal* characters and relations. The difficulty that here arises in regard to the number of respects and the remoteness of difference that are abstractly necessary for the possibility of twoness of existence exhibits more emphatically the purely dogmatic nature of the Leibnizian principle, which seems to me in any case to have no logical justification whatever[1].

[1] Much the same considerations were brought forward in the criticism of Bradley's dictum that 'distinction implies difference' (see Chapter II). If the Leibnizian and the Bradleyan principles can be in any way distinguished, it is that the former is ontological and the latter epistemological.

§ 7. Having discussed the notion of identity in its contrasts and connections with difference and otherness, we must finally examine the nature of the relation of identity itself, apart from its connections with other relations. The problem may be indicated by discussing whether identity is or is not definable. For this purpose it will be desirable to begin by considering the formula xix, rather than xiy. For xiy can only be interpreted by explicitly distinguishing the word, phrase or symbol which denotes an entity from the entity itself that is denoted; and moreover the proposition xiy (until definite equivalents for x and y are substituted) can only be adopted hypothetically or illustratively. On the other hand, the straightforward proposition xix is to be *asserted* on rational grounds for *any* value of x. Eliminating the symbol x, what is to be universally asserted is that 'Any entity is identical with itself.' Unfortunately the term *itself* can only be defined as 'what is identical with *it*'; and hence any explication of the formula seems to lead to an infinite regress. This difficulty can be removed by expressing the formula in the negative form: 'No entity is *identical* with any entity *other than* itself.' This is to be understood as a brief way of asserting: 'No entity (x) is *identical with* any entity *other than* what is *identical with* x.' This axiom expresses a universal and rationally grounded truth, expressed in terms of the two relations, identity and otherness. I shall attempt to show that the conceptions of identity and of otherness are two independent indefinables, the understanding of which is required in order intelligently to accept the truth of the above axiom or of any other proposition which explicitly or implicitly

involves the notions of identity or of otherness. Now the above axiom of identity (xix) is never explicitly used anywhere but in an abstract logical or philosophical context; on the other hand xiy is explicitly used in concrete logical and mathematical formulae; and its usage in such cases always involves the process of *substituting* y for x. The connection between identity and substitution is roughly expressed in the rule: Given xiy, we may always substitute y for x. More exactly, taking p to be any predication, 'If x is identical with y, then x is p would imply y is p.' The converse of this is also generally admitted: viz. that 'If for *every* value of p, x is p would imply y is p, then x must be identical with y.' From these two assertions conjoined it follows that the proposition 'x is identical with y' is equipollent or co-implicant with the proposition that 'For every predication p, x is p would imply y is p.' The problem thus arises whether this equipollence or co-implication can serve as a *definition* of identity. My ground for rejecting such a view is that the equipollence asserted could not be understood or utilised unless we understood what is meant by the assertions 'x is identical with x,' 'y is identical with y' and 'p is identical with p,' and had accepted these assertions as true, because of our independent understanding of what is meant by the relation of identity.

In the converse form of the identity formula we have admitted that if all the adjectives that characterise a substantive x also characterise a substantive y, then x and y are identical. But the hypothesis here is really impossible, for the adjective 'other than y' cannot characterise y. Hence there is one adjective at least

that must characterise x but not y. What then is the one relational adjective that must characterise any one existent and not any other? It must be existence itself. For to exist means to *stand out from amongst other things*. Otherness is thus presupposed by existence. In short, if the existent is what is manifested in time and space, and if time and space are wholes divisible into parts, then the only necessarily differentiating mark of one existent must be its temporal and spatial *position*. This brings us back to the Leibnizian formula, but at the expense of admitting the reality of time and space as the condition of otherness—a condition which is both necessary and sufficient.

Although in the sense explained identity always implies the legitimacy of substitution, we cannot say conversely that the legitimacy of substitution always implies identity. For whenever two predications are *co-implicative*, the one may always be substituted for the other in the same way as for substantives which are *identical*. Thus, for substantives x and y, we have the formula:

> If x is identical with y, then 'x is p' is co-implicative with 'y is p,' where p is any predication applicable to x and y.

Corresponding to this, for predications q and r, we have:

> If q is co-implicative with r, then 'q is n' is co-implicative with 'r is n,' where n is any predication applicable to q and r.

Thus, given the co-implication of the two predications (q) human being, and (r) featherless biped, we can infer that the proposition 'the number of human beings is n'

is co-implicative with the proposition 'the number of featherless bipeds is *n*.' And again given the co-implication of the two propositions (*q*) 'There is a righteous God' and (*r*) 'The wicked will be punished,' we can infer that the proposition 'That there is a righteous God is problematic' is co-implicative with the proposition 'That the wicked will be punished is problematic.' These examples show that the relation of co-implication between predications has some of the same properties as that of identification between substantives, and therefore co-implication is apt to be conceived (and even by some symbolic logicians has actually been symbolised) as equivalent to identification. It appears to me that it is theoretically possible that the conception of co-implication could be shown to correspond to factual identification; but this indeed is doubtful, because the relation of co-implication is compound, i.e. it denotes the conjunction of the two correlatives implying and implied by, whereas it seems impossible to reduce the notion of identification to the conjunction of two correlatives.

Before dismissing this subject, it must be admitted that both as regards substitution for identified substantive terms, and substitution for co-implicative predications, certain limitations seem to be required. For example Mr Russell has familiarised us with illustrations for the necessity of this limitation by such examples as that, from the identification of 'Scott' with 'the author of *Waverley*,' we cannot, by substituting the one term for the other in such a proposition as 'George IV believed Scott to have written *Marmion*' infer that 'George IV believed the author of *Waverley* to have written *Marmion*.' Or again that the number of the

apostles is identical with the number of months in the year, does not necessarily imply that anyone who doubts that the number of apostles is 12, would necessarily doubt that the number of months was 12. It appears that the only statements for which such substitutions are invalid are *secondary* propositions predicating of a primary proposition some or other relation to a thinker.

§ 8. To complete the account of identity, it is necessary to anticipate what will be elaborated in a later part of the work; and for this purpose the discussion will be restricted to identity of substantives proper, excluding any further reference to the identity of adjectives or predications. A substantive proper I have defined as what is manifested in space and time, or otherwise an existent; and the category existent has been divided into the two sub-categories called respectively occurrent and continuant. Now identity as applied to an occurrent could be illustrated thus: "The flash of lightning to which I am pointing is identical with the flash of lightning to which you are pointing.' A continuant, on the other hand, means that which continues to exist while its states or relations may be changing; identity of continuant may therefore be illustrated by some such examples as 'The body which illuminates the earth is identical with the body that warms the earth'; 'The person who was experiencing the tooth-ache is identical with the person who intends to go to the dentist.' This last conception we find discussed at some length under the heading 'Identity' by the earlier English writers Locke, Hume and Reid, who used the term to signify personal identity instead of giving to it the merely relational significance of the more modern conception. In

effect, for them the assertion or denial of identity is equivalent to the assertion or denial that there is a person who continues to exist throughout a period of time in which his various experiences may be altering in character. The destructive view represented by Hume regarded experiences as what I call occurrents, any one of which is merely replaced by another in the course of time. This destructive view is equivalent to the denial of any psychical continuant. Necessarily an occurrent experience is as such identical with itself; what Hume denied was that kind of connection between one such occurrent and another which constitutes them into alterable states of one individual self. Hence it was not the general conception of identity, but the reality of the psychical continuant that Hume denied and that Reid maintained. When this old problem is revived at the present day, it is usually formulated as the question whether there is any ultimate philosophical justification for regarding one and another experience as belonging to the same self. This phraseology, however, is misleading; for it appears to assume the existence of a self, and to raise only the question as to whether we can refer different experiences to the *same* self; whereas the real problem is whether there is a self at all. In discussions, by Hume and others, connected with that of 'personal identity,' we find the same problem raised as to the validity of the more general notion 'substance'—i.e. in my terminology, a continuant (whether psychical or physical). Now we might interpret Kant as admitting the validity of the conception of a physical continuant while denying that of a psychical continuant. On the other hand, Berkeley more obviously supported

the notion of a psychical continuant and rejected that of a physical continuant. At the present day, most philosophers who reject these notions regard them as reached by a constructive process, and therefore as being merely convenient fictions. To assume that a notion is necessarily fictitious because it owes its origin to a constructive process is fallacious, especially for those who accept the central Kantian position according to which the objective validity of any conception is established by showing it to be the result of a synthetic, i.e. constructive act of pure thought. The same relation between identity and the notion of a continuant applies to the physical as to the psychical continuant. In the physical sphere, those who reject the physical continuant maintain that any physical event is a mere occurrent which is replaced by another occurrent; and that a so-called change is merely the temporal succession of one and another occurrent. Those, on the other hand, who accept the notion of a physical continuant maintain the validity of the notion of change as distinct from mere alternation, and as therefore presupposing the conception of a physical continuant.

We may indicate more positively the distinction between the two views which respectively reject and accept the notion of a continuant, while agreeing in the application of the relation of identity. Those who deny continuance of existence as well as those who affirm it can legitimately collect all the experiences, occurring during some part or the whole of what we call an individual's total experience, to constitute a class and assert that this collection is identical with itself. On the other hand, for those who affirm the continuant, the collection is

not a mere plurality but a specific kind of unity; in other words, they hold that an intimate bond of causality subsists between the experiences attributable to one individual of a kind which does not subsist between experiences arbitrarily selected from the histories of different individuals. The notion of this unique kind of bond is, on this view, the product of a constructive act, but *not* to be dismissed on this ground as merely fictitious.

CHAPTER XIII

RELATIONS OR TRANSITIVE ADJECTIVES

§ 1. So far we have treated the adjective solely in its reference to the substantive which it characterises. We have now to consider a type of adjective whose meaning when analysed exhibits a reference to some substantive other than that which it characterises. Thus we may characterise a certain child by the adjective 'liking a certain book,' or a certain book by the adjective 'pleasing a certain child.' These adjectives predicated respectively of the child and of the book, are complex; and when we take the substantival reference out of this complex, there remains the term 'liking' or 'pleasing.' Such terms do not function as completed adjectives, and will be called relational adjectives. Propositions involving adjectives of this type may be ranged in a series according to the number of substantives to which they refer. Thus, in the following examples: 'A is wise,' 'A likes B,' 'A gives X to B,' 'A accuses B at time T of C,' the number of substantival references are respectively one, two, three and four, and the corresponding adjectives or propositions may be called monadic, diadic, triadic and tetradic.

Taking first the two-termed relation, let us consider the proposition 'X likes Y' or 'X is greater than Y.' The notion of 'X as liking Y' or of 'X as being greater than Y' is to be distinguished from the notion of 'Y as liking X' or 'Y as being greater than

X.' At the same time the thought of any assigned relation of *X* to *Y* involves the thought of a definitely assignable relation of *Y* to *X*; for example, the thought of *X* as liking *Y* involves the thought of *Y* as pleasing *X*; and *X* as greater than *Y* involves *Y* as being less than *X*. Two relational adjectives such as liking and pleasing, or greater than and less than—each of which in this way involves the other—are called correlatives, and either one is said to be the converse of the other. When the relation is expressed by a transitive verb, the opposition between active and passive expresses the mutual implication of correlatives: thus, '*X* likes *Y*' means '*Y* is liked by *X*,' or '*Y* pleases *X*' means '*X* is pleased by *Y*.'[1] Except in the case of the active and passive voice, there is no general rule of language according to which the converse of a given relative can be expressed, and therefore a special knowledge of words in current use is required in order to be able to express a relation in its converse form; as when we pass from '*X* is greater than *Y*' to '*Y* is less than *X*,' or from '*X* likes *Y*' to '*Y* pleases *X*.' However, the fact expressed in terms of a relative is the same as the fact expressed in terms of its converse, whether the terms employed are philologically cognate or not.

It must be pointed out that 'liking *Y*' or 'liking someone,' etc., is a completed adjective; and, in general, out of a relational adjective we may construct a com-

[1] Comparing 'x sleeps' or 'x is sleeping' with 'x hits y' or 'x is hitting y,' and noting that '*sleeps*' is an intransitive and *hits* a transitive *verb*, we ought properly to call *sleeping* an intransitive and *hitting* a transitive *adjective*. Thus a *relation* is properly defined as a '*transitive adjective*,' the ordinary adjective being distinguished as *intransitive*.

pleted adjective by supplementing the substantival reference. And conversely most ordinary adjectives in use can be analysed so as to elicit a relational element as a component. For instance 'amiable' contains the relational element 'liked by,' and may be roughly defined 'liked by most people.' Again, substantive words are constructed out of relational adjectives, e.g. 'a shepherd' which means 'a person who takes care of sheep.' It is noteworthy, however, that to take the substantives 'shepherd' and 'sheep' as examples of correlatives involves a double error, since the true correlatives involved in the meaning of shepherd are 'taking care of' and 'taken care of by,' which are adjectival and not substantival; while the meaning of the word 'sheep' contains no relational element at all.

§ 2. Our immediate concern will be with diadic adjectives, otherwise called coupling. Given any two substantives—say x and y—we may construct what will be termed a substantive-couple expressed by the phrase 'x to y,' which is to be distinguished from the substantive couple 'y to x.' Similarly, given any two correlative coupling adjectives—say greater-than and less-than—we construct what will be termed an adjective-couple, expressed by the phrase 'greater-than to less-than.' The significance of a substantive-couple is to be explained by defining it as that which may be characterised by an adjective-couple; and the significance of an adjective-couple, by defining it as that which may characterise a substantive-couple. Thus the relation of substantive-couple to adjective-couple is the same as that of an ordinary adjective to an ordinary substantive; and just as the latter are united through the

characterising tie, so are the former. Again, just as the extension determined by an ordinary adjective comprises the substantives of which the adjective may be truly predicated, so we may say of an adjective-couple that the extension which it determines comprises the substantive-couples of which the adjective-couple may be truly predicated. This relation between the substantive-couple and the adjective-couple is brought out by expressing the proposition 'x is greater than y' in the form

'x to y is-as greater than to less than,'

and the proposition 'y is less than x' in the form

'y to x is-as less than to greater than.'

This mode of formulation helps perhaps to explain the process of relational conversion, which may be illustrated as follows:

(1) x is greater than y,

$\therefore$ (2) x to y is-as greater than to less than,

$\therefore$ (3) y to x is-as less than to greater than,

$\therefore$ (4) y is less than x.

In passing from (1) to (2), the introduction of the term 'less than' depends merely upon knowledge of the arbitrary usage of language; but the logical validity of the step rests upon the fundamental principle of thought that every relation has its converse. Each step also requires that the order in which the adjective terms are mentioned is to be understood to correspond to that in which the substantive terms are mentioned. Similar reformulations could be applied to triadic and

higher orders of adjectives: thus 'x receives b from y will be rendered

$x : b : y$ | is-as | receiving: given by: giving to;

and 'y gives b to x' becomes

$y : b : x$ | is-as | giving: given to: receiving from;

where ':' stands for 'to,' and the ordering of the words is to be interpreted cyclically.

§ 3. The general notion of an adjective-couple that can be predicated of a substantive-couple is familiarly illustrated in what is called analogy. Take the proposition 'England to Australia is-as parent to child'; here the predicate is what I call an adjective-couple, constituted by taking the relation 'parent-of' and its converse 'child-of'; while the subject is a substantive-couple composed of the two substantives England and Australia. The copula 'is-as' marks a statement of analogy. Another example with the same adjective-couple is 'France to Algiers is-as parent to child.' From these two predications of the same adjective-couple we should infer that 'France is to Algiers as England is to Australia.' This form of proposition, however, differs importantly from that which predicates an adjective-couple of a substantive-couple. For in affirming the equivalence of the relation in which France stands to Algiers with that in which England stands to Australia, there is no indication of the kind of relation in respect of which the two substantive-couples agree. We might, in fact, compare this inference with the inference that 'X is like Y' which could be drawn from the two propositions that 'X is red' and 'Y is red'; or equally from the two propositions that 'X is square' and 'Y is

square.' We then see that such terms as similar or analogous when used to connect two substantives or two substantive-couples are quite indeterminate with respect to the ground of similarity or analogy; so that from the assertions 'A is similar to B' and 'B is similar to C' it could never be inferred that 'A is similar to C.' My formulation of the relational proposition 'A to B is-as p to q' is of course suggested by the arithmetical expression for a ratio and the equality of ratios, but here there is a danger of misunderstanding; for in my phraseology we could assert that '8 to 6 is-as greater by 2 to less by 2,' and again that '5 to 3 is-as greater by 2 to less by 2' and hence that '8 to 6 is-as 5 to 3': but in arithmetic the phraseology 'p to q' is understood to denote exclusively a relation under the genus called *ratio*, and could not be applied to a relation under any other genus such as difference or distance, etc., etc. Instead of calling such a term as ratio, difference or distance, etc., by the familiar name *genus*, it ought, properly speaking, to be termed 'relational determinable' in contradistinction to three-fifths, or minus two, or three yards distant, which are 'relational determinates' under their respective relational determinables. Whereas *here* the phraseology 'a to b is-as p to q' is used for relational determinates under *any* relational determinable, in arithmetic this phraseology is limited to relational determinates under the one relational determinable called *ratio*. It must be observed that, when the term analogy is explicitly referred to a given relational determinable, such as *ratio*, then, from the assertion that two substantive couples are analogous to the same substantive couple, we may infer that the

two are analogous to one another. This is precisely parallel to the case of agreement or similarity when explicitly referred to any given adjectival determinable, such as colour.

§ 4. This account of relational adjectives leads to a consideration of a species of tie distinct from the characterising tie, which we shall call the *coupling tie*. In the phrase 'x to y' the word *to* has been chosen to indicate this tie, and hence the effect of the coupling tie is to construct a substantive-couple. Any preposition or prepositional phrase such as *of*, *by*, *for*, *at*, *with*, *in*, *in reference to*, indicates the presence of the coupling tie. We must not, however, in general say that the preposition denotes merely a tie; for a difference of preposition often stands for a difference in the relation predicated. For example 'x is influenced to move *towards* y' has a different meaning from 'x is influenced to move *away from* y'; where the difference of preposition is seen to entail a difference of relation—namely the difference between attraction and repulsion. In fact, prepositions used along with adjectives or verbs express determinate *modifications* of relation. The essential feature of a tie, on the other hand, is that it is incapable of modification, and hence we frequently find that it does not enter as a separate verbal component in a sentence.

Whenever a tie (whether it be the characterising tie, or the coupling tie, or any other) does not appear as an actual word, there are conventions of language which indicate its presence. In languages in which inflexion is largely used, such as Latin and German, there are two main kinds of grammatical rule; namely, the rules

of concordance and the rules of governance. We shall find that the rules of concordance point to the presence of the characterising tie; and those of governance to the coupling tie. The rules of concordance are, briefly, that adjectives and verbs must agree in gender, number and case, with the substantives that they characterise; so that the characterising tie is not necessarily expressed by use of the verb 'to be' but merely by inflexion. On the other hand, the rules of governance always determine the case—genitive, dative, accusative, or ablative—of the substantive that is introduced along with any transitive verb, relational adjective, or preposition. We find, especially in Latin, that considerable changes in the *order* of words (which may vary for purposes of rhetorical significance) are permissible because of the inflexions which are understood to indicate (i) by grammatical *agreement*, how the words are to be attached in thought by the characterising tie, and (ii) by grammatical *governance*, how they are to be attached in thought by the coupling tie. Furthermore, where modification of case occurs (with or without a preposition), not only the coupling tie, but also the special modification of relation that is to be understood is grammatically indicated. The characteristic of English, in contrast to highly inflexional languages, is that no inflexions are required by rules either of concordance or of governance, except in the two instances: (i) for differences of person and number in many verbs (which illustrate the *characterising* tie), and (ii) the accusatives—him, her, me, us, them, whom (which illustrate the *coupling* tie). All the other instances of inflexion in English—for example the possessive pronouns and the

tenses of verbs—are used, not according to any rules of concordance or governance, but to express distinctions of meaning. The difference between the two kinds of inflexion—the one being significant and the other syntactic—is brought out by comparing the English 'her father' or 'his mother,' where the difference of gender is significant and not syntactic, with the French 'son père' or 'sa mère,' where the difference of gender is syntactic and not significant. In English, the conventional rules of concordance or governance are replaced—except in the two cases mentioned above —by the equally conventional *ordering* of the words.

§5. The coupling tie—which might have been called the prepositional tie, in consideration of the grammatical rules of governance, or again the relational tie, in consideration of the philosophical problems that have been raised in regard to the nature of relation—is of fundamental importance in discussing one of the paradoxes that Mr Bradley and others have found in the general notion of relation. The paradox is briefly brought out in the following contention: when we think of x as being r to y, we have first to relate x to y by the relation r, and then relate the relation r to x by—say r'— and r to y by—say r'', another relation. This again will require that x should be related to r' by the further relation r''', which will lead to an infinite regress on the side of x, and a similar regress on the side of y. This paradoxical contention is met by pointing out that in constructing an object out of the constituents x, r, and y, we do not introduce another constituent by the mere act of constituting these constituents into a unity. The pretence of paradox is due to the assumption that to

the act of relating or constructing there corresponds a special *mode* of relation; so that a tie is confused with a relation. That a tie and a relation are distinct is brought out by considering the fact that if, for a given adjective—whether ordinary or relational—we substitute another adjective, we shall have constructed a *different* unity; but, if we drop the characterising tie with a view to replacing it by some adjective or relation, then either the unity itself is destroyed, or it will be found that the characterising tie remains along with the adjective or relation so introduced. Similarly, the coupling of terms is not a *mode of relating* them for which another mode of relation could be substituted; for, if they were uncoupled, again the unity would be destroyed.

The distinction between a tie and a relation may be brought out from another point of view by the consideration that the specific difference between one kind of tie and another is determined by the logical nature of the constituents tied. Thus the use of an adjective in general involves the characterising tie, by which it is attached to a substantive; and the use of a relational adjective in particular further involves the coupling tie by which the two substantive-terms are attached to one another. On the other hand, where terms are related by a genuine relation, their logical nature allows any specific relation to be replaced by any other, this other being in general under the same relational determinable.

Now just as we must distinguish between any relation and the relational (or coupling) tie, so I have throughout assumed, and it is of the first importance to emphasise, the distinction between characterisation and

the characterising tie. I have, in fact, spoken repeatedly of the *relation* of adjective to substantive; and this is the relation called characterisation—a specific kind of relation to be distinguished, for instance, from 'liking' or 'exceeding,' etc. While characterisation, then, is a relation, the characterising tie (like any other tie) is a mode of *connection*, represented usually by the participle 'being.' This participle may be expanded into 'being-characterised-as-being,' and it will still represent merely the characterising tie. Thus the following series of examples are seen to be but different modes of expressing precisely the same fact: I am thinking (1) of this |as| tall; (2) of this |as-being| tall; (3) of this |as-being-characterised-as-being| tall; where the phrase enclosed in vertical lines represents nothing more nor less than the characterising tie. The equivalence between 'S as-being P' and 'S as-being *characterised* as-being P,' etc., is precisely analogous to the arithmetical equalities: '$S \times P$' = '$S \times 1 \times P$' = '$S \times 1 \times 1 \times P$,' where S stands for any magnitude and P for any numerical multiplier, while the number *one* takes the place of the relation '*characterised*,' and the operator '$\times$' takes the place of the tie 'as-being.' We will therefore call 'characterisation' the *unit* relation, because it may be conceived as a factor in every adjective and in every relation. The conception of 'characterised' as a relational factor, analogous to *one* in the expression $S \times (1 \times P)$, will be shown more precisely by adopting a different mode of bracketing (3) whereby it becomes: (4) I am thinking of 'this |as-being| characterised-as-being-tall.' Now this last expression is formally analogous to (5) I am thinking of this |as-being| taller-than-that. The phrase

|as-being|, occurring both in (4) and (5), represents the characterising tie; but in (5) the relation (with its tie) is expressed by 'taller-than,' and supplemented by the substantive-term 'that,' while in (4) the relation (with its tie) is expressed by 'characterised-as-being' and supplemented by the adjective-term 'tall.' Since, moreover, we may expand the tie in (5) into 'as-being-characterised-as-being,' we see that 'characterised' is a latent factor in every relation, as well as in every ordinary adjective. Thus 'characterised' is a latent factor even in the relation 'characterising': for 'I am thinking of P as characterising S' may be expanded into 'I am thinking of P |as-being| characterised as characterising S.' This reformulation incidentally shows that, though *characterised* has the properties of a unit relation, yet its converse *characterising* has not these properties.

§ 6. Now the form of proposition in which 'characterised' is introduced explicitly as a relation, derives its significance and its legitimacy from our having taken an adjective—namely 'tall'—as a term. We are therefore extending the application of the notion of a relation, when in this way we take an adjective as *term*, instead of (as hitherto) a substantive. Indeed no limit can be imposed upon the kind or category of entity which may constitute a 'term' of which adjectives or relations to other entities may be predicated. In particular, we must recognise that certain adjectives may be significantly predicated of adjectives and of propositions, and even of relations; and that certain relations may be significantly predicated as subsisting between a substantive and an adjective, or between one adjective and another, or between one proposition and another, or even between

one relation and another. The adjectives, relations, or propositions of which other adjectives or relations may be predicated must when so connected be called *terms*, in contrast with the adjectives or relations predicated of them. The logical mode in which adjectives, relations or propositions enter as *terms* into a construct, is reflected in language by the substantival form assumed by them; e.g. intolerance, hatred, the enthusiasm of the people, that matter exists, to be or not to be, etc. etc.

Underlying the merely nominal question whether adjectives, relations or propositions when functioning as substantives in a construct should be *called* terms, two philosophical questions arise, which I shall here deal with rather summarily. First, is it literally the same entity which can be treated indifferently *either* as an adjective in its primary or natural functioning, *or* as a quasi-substantive of which certain other adjectives or relations may be predicated? To this I give an affirmative answer; and the objections to my view are (I think) met by insisting that the adjectives or relations which may be significantly predicated of *primary* adjectives or relations (as they may here be called) belong to a different logical sub-category from these latter, and may be called *secondary*. Thus, it is not that the primary adjective changes its category when functioning as quasi-substantive, but it is that the secondary adjective must be said to belong to a special sub-category, determined by the category of the primary adjective of which it may be predicated. But a second question arises: whether the relation or adjective that is *apparently* predicated of a relation, adjective or proposition, is *really* so predicated; or whether it is pre-

dicated rather of certain substantives to which there is implicit or explicit reference. As regards this question, it must be pointed out that while relations, subsisting primarily between certain entities entail relations between other entities involved in or connected with the former, yet the relations thus entailed are not identical with the primary relation; so that whether a relation is predicated of this or of that kind of entity needs separate discussion in each type of case. A typical example will be discussed in a succeeding paragraph.

§ 7. We now propose to analyse relational propositions of more complicated forms. The principles (previously illustrated) in accordance with which a diadic proposition may be reduced to a monadic by hyphening (or bracketing) the predicate, and may be converted by substituting for the given relation its correlative, may be extended to relational propositions of higher orders. In such propositions any one of the substantive-terms may be taken as subject, and a complex containing the remaining substantive-terms as monadic predicate; or any couple of substantive-terms may be taken as subject and a complex containing the remaining substantive-terms as diadic predicate; and so on. In such transformations, any *permutations* that are made amongst the substantive-terms, will require the substitution of 'cognate' modes of expressing the given relation; and, as in diadic relations, the order of relationality is reduced by constructing a complex predication indicated by a hyphen or bracket. For example, the triadic proposition '*A* gives *X* to *B*' is reduced to a diadic by transforming it into '*A* gives-*X* to *B*' or '*B* is receiver-from-*A* of *X*' or '*X* is given-to-*B* by *A*,' which again may be converted,

respectively, into 'B receives-X from A,' 'X is given-by-A to B,' 'A is giver-to-B of X.' Thus the single radical relation represented by such a verb as 'to give,' from which a triadic proposition is constructed, gives rise to three pairs of converse forms (making six correlatives), namely: giver-to and receiver-from; receiver-of and given-to; given-by and giver-of. And, furthermore, each of the above six diadics becomes monadic by completing the hyphening of the predicate without any further verbal alteration.

In the above illustrations a *single* radical relation was involved; but another kind of complication arises when, besides a number of *substantival* components, the proposition contains more than one *adjectival* or *relational* component, these being in general indicated by different verbs. For example, the proposition 'A prevented B from hurting C' contains the three substantive-terms A, B, C, together with two verbs, of which 'prevent' is the principal and 'hurt' is sub-ordinate. Such a proposition, treated merely as triadic, may be transformed by permutation into (e.g.) 'B was prevented by A from hurting C,' and by bracketing into 'C was saved-from-being-hurt-by-B by A.' But another form of analysis may be used where—as in the case before us—we are dealing, not only with a plurality of substantive terms, but also with a plurality of radically different relational components. In this analysis, we take as a bracketed constituent a complex containing all the components of a complete proposition; viz. the proposition that contains the *subordinate* relation or adjective. A proposition in this aspect may be called a *possibile*. The *possibile* in our illustration is 'B's-hurting$C$,' and

the proposition predicates of the person *A* the relation 'preventing.' The distinction between the two modes of bracketing—in both of which a triadic proposition is reduced to a diadic—will be shown by comparing: '*A* is | preventer-from-hurting-*C*| by *B*' with '*A* |prevents| the-hurting-of-*C*-by-*B*.' In the former the *terms* of the relation are the two persons '*A*' and '*B*,' and the relation may be expressed:

> '*A*' *to* '*B*' *is-as* 'preventer-of-hurt-to-*C*-from' *to* 'prevented-from-hurting-*C* by';

in the latter the *terms* are the person '*A*' and the *possibile* '*C*'s being hurt by *B*,' the relation being:

> '*A*' *to* '*C*'s being hurt by *B*' *is-as* 'preventer-of' *to* 'prevented-by.'

The immediate purpose of this illustration is to show that a proposition that predicates a relation of a term to a *possibile*, entails also the predication of certain complex relations between this principal term and the various substantive constituents of the *possibile*. The example under consideration may be more fully expressed by explicit reference to the action of *A* which was causally preventive: thus, '*A*'s-instructing-*D* prevented *B*'s hurting-*C*.' This more explicit form of statement is typical of the causal proposition as such which, properly speaking, always relates one *possibile* to another either by way of production or of prevention. For example, such a statement as 'The earth causes the fall of the stone' is an elliptical expression for the more fully analysed proposition 'The-proximity-of-the-earth causes the-fall-of-the-stone.' Since, however (as follows from the above remarks), a diadic relation of one *pos-*

sibile to another entails certain more complex relations amongst the substantive constituents of the possibilia, it is formally legitimate to assert that the earth stands in *some* relation to the fall of the stone, or again that the earth stands in *some* relation to the *stone*, though each of these relations is more complex than the diadic relation of causation, which holds between the two possibilia.

§ 8. The above discussion leads to the special problem of the nature of the relation of assertion. Consider the simplest case: '*A* asserts *S* to be characterised by *P*.' This contains the three terms *A*, *S* and *P*, together with the two verbs or relations 'assert' and 'characterise,' of which 'assert' is the principal, and 'characterise' the subordinate. Taking the complex as expressing a triadic relation of *A* to *S* to *P*; it may be reformulated in several ways: such as, '*A* asserts-*P*-to-characterise *S*'; '*P* is asserted-by-*A*-to-characterise *S*,' which predicate diadic relations of *A* to *S*, and of *P* to *S* respectively. But the more natural mode of expressing the relation is as one of the thinker *A* to the *possibile* '*S* being (characterised by) *P*.' My account of the relation of causation as holding primarily between possibilia but also as entailing relations (of a higher order than diadic) amongst the component terms of the possibilia, must be applied to any such relation as that of interrogating, doubting, considering, affirming, denying, etc., in which a thinker may stand to a *possibile* or *assertibile*. The relation of the thinker to the proposition as a whole does not preclude—it rather *entails*—relations to the constituents of the proposition. Thus, the relation of the thinker to the subject *S* is expressible in terms of *P*,

viz., 'asserts-P-to-characterise'; and that of the thinker to the predicate P is expressible in terms of S, viz., 'asserts-S-to-be-characterised-by.'

This principle—that assertion or judgment involves a relation of the thinker to each of the constituents of the proposition as well as to the proposition as a unitary whole—is familiarly (but misleadingly) expressed in some such form as that 'in judgment the thinker asserts a relation of one idea to another.' If by idea was really meant 'the object of a thought' rather than 'the thought of an object,' this statement would, in my view, be essentially correct. Or more briefly: I agree that judgment relates the object of one thought to the object of another thought; but I deny that judgment relates the thought of one object to the thought of another. That is to say, though judgment requires us to think about objects, the judgment is *about* these objects, and not about our thinking about them. The account of judgment that is to be rejected involves a confusion between a primary proposition which is about objects, and a secondary proposition (as it may here be termed) about our ideas of objects; and, if this identification of a primary with a secondary proposition is consistently carried out, we should have to interpret a secondary proposition as a tertiary; namely, as being about our ideas about our ideas of objects, and so *ad infinitum*. But admitting, as I have done, that in judgment we assert a relation of one object of thought to another, say of S to P, it is necessary further to consider what sort of *idea* we have of S or of P when we judge that 'S is P'; and here I propose to distinguish between the specific and the generic idea of S or of P. The *specific* idea of S that is involved

in doubting or asserting the proposition, is the idea of S as 'characterised by P'; and this idea includes the *generic* idea of S as 'characterisable,' i.e. as of the nature of a substantive. Similarly the *specific* idea of P is the idea of P as 'characterising S' and this includes the *generic* idea of P as 'characterising'; i.e. as of the nature of an adjective. But here it is to be observed that the *relation* that may be said to be predicated, viz. that of characterisation, does not subsist between the idea of S and the idea of P, since each of these ideas is specifically completed in the single complex idea of 'S-as-characterised-by-P' or of 'P-as-characterising-S' or again of 'the-characterisation-of-S-by-P'; and I hold that these three phrases express different modes of *constructing* one and the same *construct* or complex object of thought. No difference of principle is involved when dealing with an explicitly material relational proposition, such as 'S is $\hat{R}$ to T.' Here the *specific* idea of S is the completed idea of S as 'being $\hat{R}$ to T'; that of T is the completed idea of T as 'being $\check{R}$ to S'; that of $\hat{R}$, as 'relating S to T'; and that of $\check{R}$ as 'relating T to S.' These *specific* ideas include such *generic* ideas as that of S and of T as being substantives and of $\hat{R}$ and $\check{R}$ as being a pair of relations correlative to one another. As in our simpler illustration, the completed idea is of a complex object of thought constructed out of three constituents (S, R, T) bound together in a certain form of unity.

CHAPTER XIV

LAWS OF THOUGHT

§ 1. It has been customary to apply the phrase Laws of Thought to three specific formulae; but the application of the phrase should be extended to cover all first principles of Logic. By first principles we mean certain propositions whose truth is guaranteed by pure reason. It is often too hastily said that logic as such is not concerned with truth but only with consistency; as if a conclusion were guaranteed by formal logic merely because it is consistent with any arbitrarily assumed premisses. But this entirely misrepresents the function of formal logic, which is not permissive, but rather prohibitive. It guarantees the truth—not of any proposition that is consistent with the premisses—but only of the proposition whose contradictory is inconsistent with the premisses. And even this statement goes too far; for logic does not allow any arbitrarily chosen premisses to be taken as true; and thus the only conclusions that it can be said in any sense to guarantee are those which have been correctly inferred from premisses that are themselves true. When consistency is placed in a kind of antithesis to truth, it seems often to be assumed that logic is indifferent to truth. That the reverse is the case is shown by the consideration that to say that a conclusion is validly drawn from given premisses is tantamount to asserting the truth of a certain composite proposition, viz. that the premisses imply the conclusion.

In enunciating and formulating the fundamental principles of Logic, we shall not enter into the question whether they are all independent of one another, nor into the problem as to how a selection containing the smallest possible number could be made amongst them from which the remainder could be formally derived. This problem is perhaps of purely technical interest, and the attempt at its solution presents a fundamental, if not insuperable, difficulty: namely, that the procedure of deriving new formulae from those which have been put forward as to be accepted without demonstration, is governed implicitly by just those fundamental logical principles which it is our aim to formulate explicitly. We can, therefore, have no assurance that, in explicitly deriving formulae from an enumerated set of first principles, we are not surreptitiously using the very same formulae that we profess to derive. If this objection cannot be removed, then the supposition that the whole logical system is based on a few enumerable first principles falls to the ground.

§ 2. The charge has been brought against all the fundamental principles of Formal Logic that they are trivial; or otherwise that they are nothing but truisms. Now a truism may be defined as a proposition which is (1) true, and (2) accepted by everybody on mere inspection as true; and these are just the characteristics required of a fundamental principle of logic. Hence to charge the fundamental formulae with being mere truisms is not to condemn them, but to admit that they are fitted to fulfil the function for which they are intended. This function is to enable us to demonstrate further formulae, some of which, though true, are not accepted by every-

body on mere inspection as true. It is an actual fact that by means of truisms and truisms alone we can demonstrate truths which are not truisms. The above and similar criticisms directed against the fundamental formulae of Logic will be best met by directly examining this or that formula so as to bring out its precise significance in view of the different points of view from which it has been criticised; and we shall adopt this plan as occasion offers.

§ 3. Before enunciating the fundamental principles in detail, we will enquire into what is implied in speaking of them as 'Laws.' The word *law* is closely connected with the notion of an imperative; and many logicians of the present day hold that the so-called laws of thought are no more imperatives than are the axioms of arithmetic or geometry. With this view I agree, inasmuch as the axioms of mathematics can themselves be regarded as having an imperative aspect; but this is because all truth may be so regarded. The idea of truth and falsity, in my view, carries with it the notion of an imperative, namely of acceptance and rejection—a corollary from the theory which insists on the reference of judgment and assertion to the thinker. For it is only so far as assertion is recognised to be a mental act, that the notion of an imperative becomes relevant. An imperative of reason implies a restraint upon the voluntary act of assertion—a restraint which does not, however, infringe the freedom that characterises every volition, since the obligation to think in accordance with truth is self-imposed. Any study of which imperatives constitute the subject-matter has been called a *normative* science, and normative sciences have

been contrasted with positive sciences. But from a certain point of view every science may be said to exercise an imperative function in so far as any mistake or confusion in the judgments of the ordinary man is corrected or criticised by the scientist as such. Every science therefore can without any confusion of thought be regarded as normative; which is only another way of saying that the notions of truth and falsity as predicable of propositions carry with them the notions 'to be accepted' or 'to be rejected' understood as imperatives. But an explanation can be given for the restricted use of the term normative to logic, aesthetics and ethics: viz., that, while each deals with a certain kind of mental fact, it does not deal with it merely *as* fact. Every science which deals with man, either in his individual or social capacity, takes as its topic the description of mental facts—including an analysis of how men think, feel and act; but such a descriptive study of our thoughts, feelings and actions (including their causal relations) treated generally, historically or speculatively, is to be distinguished from the study of precisely the same facts in relation to certain norms or standards, and from the critical examination of these norms or standards themselves. The division of *sciences in general* into normative and positive is, therefore, unsound, inasmuch as *all* sciences may be regarded as normative in the sense that they are potentially corrective of mistaken, false or obscure views. This division (into normative and positive) is therefore properly restricted to sciences dealing with psychological material; thus the positive or descriptive treatment of mind—in its thinking, feeling or acting aspect—is (like all sciences) normative in the sense of

being potentially corrective of false judgments on the topics directly dealt with; while the treatment in Logic, Aesthetics and Ethics of these same processes is normative in the more special sense that these sciences examine and criticise the norms of thought, feeling or action themselves. Within the range for which the antithesis between normative and positive holds, the distinction between a descriptive or causal account of psychological or sociological matters, and an examination of standards or norms, is now-a-days of the first importance, inasmuch as the substitution of *causal description* in the place of *evaluation of standard* has been woefully common in works which profess to found Ethics upon psychology or sociology.

§ 4. To return to the consideration of the principles which exercise an imperative function. The fundamental formulae for conjunctive and composite propositions have been given in the chapter on compound propositions; these must be included in the general consideration of the Laws of Thought. Certain of these laws, and in particular the Reiterative, Commutative and Associative laws of Conjunction are—not only the materials which explicitly compose the logical system—but are also implicitly used in the process of building up the system. Thus, for example, we not only explicitly formulate the Reiterative Law, but in making repeated use of this or of any other law, we are implicitly using the Reiterative Principle itself. This will be seen to hold in the same way of the Commutative and Associative principles of Conjunction. Finally, inasmuch as the system is developed by means of inference, the essential principles of implication are not only *explicitly*

formulated in the formulae for composite propositions, but also *implicitly* used in constructing the logical system itself.

§ 5. The next set of laws to be considered will be those which express the nature of *Identity*, since this is a formal conception which applies with absolute universality to all possible objects of thought whatever the category to which they may belong. Identity is a relation, and as such has certain properties which are exhibited in what we shall call the *Laws of Identity*. Relations in general may be classified according to the formal properties they possess, irrespectively of the terms related. It will be necessary here to introduce and define three of these properties, viz., transitiveness, symmetry and reflexiveness. Using the symbols x, y, z, to stand for the terms of any relation, and the symbols $\hat{r}$ and $\check{r}$ for any relation and its converse, then,

(1) the relation $\hat{r}$ is called transitive: when 'x is $\hat{r}$ to y' and 'y is $\hat{r}$ to z' together implies 'x is $\hat{r}$ to z' for all cases of x, y, z; for example, *ancestor*, *greater than*, *causing*, *implying*;

(2) the relation $\hat{r}$ is called symmetrical: when 'x is $\hat{r}$ to y' implies 'y is $\hat{r}$ to x'; in other words, when 'x is $\hat{r}$ to y' implies 'x is $\check{r}$ to y,' for all cases of x and y; for example, *cousin*, *incompatible with*, *other than*;

(3) the relation $\hat{r}$ is called reflexive: when 'x is $\hat{r}$ to x' for all cases of x; for example, *compatriot of*, *simultaneous with*, *homogeneous with*.

Now the three Laws of Identity are most simply expressible by the statement that identity is (1) transitive, (2) symmetrical, (3) reflexive; or otherwise, for

every object of thought (represented by the symbols x, y, z).

(1) *Transitive Law*: If x is identical with y, and y is identical with z; then x is identical with z.

(2) *Symmetrical Law*: If x is identical with y, then y is identical with x.

(3) *Reflexive Law*: x is identical with x.

It will be observed that there are a host of other relations which have these same three properties; e.g. contemporaneous, homogeneous, compatriot, numerically equal, equal in magnitude, etc., but analysis of every such relation shows it to contain a reference to some identical element, upon which these formal properties depend.

§ 6. The phrase 'Law of Identity' has been traditionally used for one of the three fundamental logical principles, known as the Laws of Identity, of Non-Contradiction, and of Excluded Middle, to which the term 'Laws of Thought' has been usually restricted; but, since these three laws relate exclusively to *propositions*, whereas the conception of identity applies to all objects of thought, I propose to substitute for the traditional terminology, the Principles 'of Implication,' 'of Disjunction' and 'of Alternation' respectively; and to insert a fourth, to be called the 'Principle of Counter-implication.' The four together will be entitled 'the Principles of Propositional Determination.' The four laws are thus brought into line with the four forms of composite proposition discussed in a preceding chapter. The composite propositions expressed in their general form, i.e. in terms of two *independent* components p, q, are of course not guaranteed as true by pure logic; in

other words, they require material or experiential certification as opposed to merely formal or rational certification. The *principles*, on the other hand, are those cases of the composite propositions, expressed in their quite general form, the truth of which is guaranteed by pure logic. For the purposes of formulating the principles on the lines of the four composite functions, we may slightly modify the expression of these latter as follows:

(1) Implicative Function:	If P is true, then Q is true.
(2) Counterimplicative Function:	If P is false, then Q is false.
(3) Disjunctive Function:	Not both P true and Q true.
(4) Alternative Function:	Either P true or Q true.

The *principles* are obtained by substituting P for Q in the implicative and counterimplicative functions, and P-false for Q-true in the disjunctive and alternative functions. Thus:

Principles of Propositional Determination

(P being any proposition)

(1) *Implicative*:	It must be that if P is true, then P is true,
(2) *Counterimplicative*:	It must be that if P is false, then P is false,
(3) *Disjunctive*:	P cannot be both true and false,
(4) *Alternative*:	P must be either true or false,

where the words 'must be' and 'cannot be' serve to indicate that the principles are formally or rationally certified.

This formulation uses a single proposition P together with the two adjectives true and false, in preference to the more usual mode of expression which employs two propositions, P and not-P, and a single adjective 'true'; as in the following:

(1) If P is true, then P is true.

(2) If not-P is true, then not-P is true.

(3) P and not-P cannot both be true.

(4) Either P or not-P must be true.

There are several reasons for adopting the former of these two modes of formulation in preference to the latter. In the first place it uses the comparatively simple notion of P being false instead of the rather awkward notion of not-P being true. Secondly it enables us to define 'contradiction' by means of the principles, which would be impossible without a circle if we introduced the contradictories P and not-P into the formulation. In the third place, the introduction of the phrases P-true and P-false is in accordance with the fact that the adjectives true and false are the first characteristics by which the nature of the proposition as such is to be understood. A closer analysis of this formulation of the alternative and disjunctive principles will throw further light on the nature of the antithesis between the adjectives true and false. We have emphasised the point that these adjectives are predicable only of propositions; in other words 'anything that is true or false is a proposition'; the principle of alternation adds to this statement its complementary, viz., 'anything that is a proposition is true or false.' It is clear, of course, that these two statements are not synonymous. Again, the principle of disjunction states that the adjectives true and false are incompatible; and this again goes beyond what is explicitly involved in the statement that they are predicable exclusively of propositions.

The most obvious immediate application of these

principles is obtained by taking P to stand for a definite *singular* proposition: 's is p,' where 's' stands for a uniquely determined or singular subject, and 'p' for any adjective. Then P-false becomes 's is not-p.' In this application, the four principles may be called the Principles of Adjectival Determination, and assume the following form:

Principle of Implication:	If s is p, then s is p.
Principle of Counterimplication:	If s is not-p, then s is not-p.
Principle of Disjunction:	s cannot be both p and not-p.
Principle of Alternation:	s must be either p or not-p.

In this application, the principles are expressed in terms of any adjectives p and not-p predicated of any subject s; instead of being expressed in terms of the adjectives true and false predicated of any proposition P. In ordinary logical text-books the 'Laws of Thought' are almost always expressed in this specialised form; but, by this mode of enunciation, the *generality* which characterises the formulation in terms of propositions is lost; for when 'adjectives predicated of any subject' is substituted for 'propositions,' we have only a special case from which the general could not have been derived. It is convenient for many purposes to use the term 'predication' to stand for 'adjective' *or* 'proposition'; thus we may include both the general and the special formulae of determination in the abbreviated forms 'If p then p'; 'If not-p then not-p'; 'Not-both p and not-p'; 'Either p or not-p'; where p stands for any *predication*. The two sets of formulae might again be expressed—without any modification of meaning—in the form of universals, since P stands for *any* proposition, and s for *any* subject; thus:

Generalised Form for Propositional Determination	Generalised Form for Adjectival Determination
If any proposition is true, it is true	If anything is p, it is p
If any proposition is false, it is false	If anything is not-p, it is not-p
No proposition can be both true and false	Nothing can be both p and not-p
Any proposition must be either true or false	Anything must be either p or not-p

A comparison between these two generalised formulations of the principles will bring out the important distinction between false and not-true and again between true and not-false. According to the principles of adjectival determination 'Any*thing* must be either true or *not-true*'; whereas of propositions we can say 'Any *proposition* must be either true or *false*.' Now, since it is only propositions of which truth is properly predicable, therefore of anything that is not a proposition the adjective true must be denied; thus we must say 'The table is not true' on the elementary ground that 'the table' is not a proposition; but we cannot say that 'The table is false,' because it is only propositions which can be said to be false. Thus the principles of propositional determination force upon us the notable consideration that the word false does not really mean the same as not-true. To have expressed the principle of alternation in the form 'Anything must be either true or false' without the necessary restriction to a proposition would have been actually wrong. On the other hand, the form 'Any proposition must be either true or not-true' is not sufficiently determinate, for this alternative would hold of any subject whatever and fails to express the alternative peculiar to the proposition itself.

On the same ground, the disjunctive principle is not properly expressed in the form 'No proposition can be both true and not-true.' This affords another, and, in my view, the most important justification for formulating the principles in terms of the adjectives true and false instead of in terms of the propositions *P* and not-*P*. In illustration of the above discussion we may point to the analogy between the four adjectives true, false, not-true, not-false and the four adjectives male, female, not-male, not-female. The antithesis between male and not-male or again between female and not-female is applicable to any subject whatever, but that between male and female is applicable exclusively to organisms. Analogously the antithesis between true and not-true or again between false and not-false is applicable to any subject-term whatever, but that between true and false is applicable exclusively to propositions.

§ 7. Now the so-called Law of Identity—which I have expressed in the form 'If *P* is true then *P* is true' where *P* stands illustratively for any proposition—has been the favourite object of attack by critics of the principles of formal logic, on the score of its insignificance or triviality. The reason why the formula appears to have little or no significance is that its implicans is *literally* identical with its implicate. It will be found, however, that the necessary condition for the explicit use of the relation of identity is that the identified element should have entered into different contexts. It must be noted that this necessary reference to difference of context does not render the relation of identity other than absolute, i.e. it in no way implies that in its two occurrences the identified element is partly identical

and partly different. The application of this general condition to the Principle of Implication requires us to contemplate the proposition 'P is true' as one that may have been asserted in different connections or on different occasions or by different persons. Then, since the formula 'If P is true then P is true' is to be understood as logically general, its full import can be expressed in the form: 'If the asserting of P in any one context is true, then the asserting of P in any context whatever is true.' If this analysis be accepted, it will be found that the principle could not have been enunciated except for the possibility of identifying an assertum or proposition as distinct from the various attitudes (belief, interrogation, doubt, denial) which might have been adopted towards it on different occasions by the same or different persons. One important element of meaning, therefore, implicit in the formula is that it tacitly implies the identifiability of a proposition as such.

Turning now to the adjective 'true' as it occurs in our analysis of the formula, let us contrast it with certain adjectives that are predicable of things in general. The principle—that what can be asserted in one context as true must be asserted in any other context as true—is more familiarly particularised in the form 'any proposition that is once true is always true'; that is to say that 'true' as predicable of any proposition is unalterable; whereas there are certain adjectives and relations predicable of things in general which may characterise them only temporarily. Contrasting, for instance, the Principle of Propositional Determination 'If any proposition is true it is true' with the Principle of Adjectival Determination 'If anything is p it is p,'

we find that in the former the copula 'is' is to be interpreted without reference to the present or any other assigned time; whereas in the latter the adjective *p* may be alterable, so that the copula 'is' must here be understood as referring to definitely assigned time. In the case of anything that is at an assigned moment of time *p*, the principles of logic do not entitle us to assert that it will be or has always been *p*. Taking as examples 'The water has a temperature of 30° C.' or 'Mr B. is at home,' we must say on the one hand that if these propositions are true at any time, they are true at all times. But we must not say that if the predicate 'having a temperature of 30° C.' or the relation 'being at home' is true of a given subject at one time, it will be true at all times. This obvious comment would not have been required if language had distinguished in the mode of the verb 'to be' between a timeless predication and a tense (present, past or future). Certain logicians have, however, deliberately denied the dictum that what is once true is always true, and their denial appears to be due to a confusion between the time at which an assertion is made, and the time to which an assertion refers; or as Mr Bosanquet has neatly put it—'between the time *of* predication and the time *in* predication.' Others, i.e. the Pragmatists, have made the denial of this dictum a fundamental factor in their philosophy, inasmuch as they have taken the term 'true' to be virtually equivalent to 'accepted,' whereas everybody else would agree that the term is equivalent rather to the phrase '*to be* accepted.' Again, the dictum would not have been confidently admitted in the days before the principles of Logic had been formulated by Aristotle, when the

antithesis between the immutability of truth and the mutability of things appears to have presented an insurmountable problem. Since, then, it has been disputed from three different points of view that the truth or falsity of a proposition is independent of the time of assertion, the first Law of Thought—my interpretation of which brings out clearly this quality of truth—is effectively freed from the charge of triviality.

But not only must we interpret the principles as implying the unalterability of truth, for further, according to the principle of disjunction, a proposition cannot be both true and false; and this is to be interpreted to imply that, if a proposition is true in any one sense, there can be no sense of the word true in which it could be false, or other than true. On this interpretation the principle would be opposed by those philosophers who employ the words relative and absolute, or similar terms, to distinguish two kinds of truth. In consistency with this philosophical position, the term 'true' must be said to have two meanings, so that one and the same proposition might be true in one meaning of the term 'true' and false in another. It would seem that it is only on this theory that philosophers could maintain that a certain proposition such as 'Matter exists' is true in or for science, and at the same time false from the point of view of philosophy. According to the view, however, of those who maintain rigidly the validity of the Principles of Determination, it cannot be said that the same proposition is true in one sense and false in another sense, although it may be said, of course, that one sense given to a certain collocation of words would yield a true proposition, while another sense given to the same

collocation of words would yield a false proposition. We must deny for instance that 'Matter exists' can be true in one sense and false in another sense, though we do not for a moment dispute that 'Matter exists' in one sense may be true while 'Matter exists' in another sense may be false. It is noteworthy that the confusion here is exactly parallel to that between the time of predication and the time in predication. Thus the assertion 'Mr Brown is at home' cannot be true at one time and false at another; though that 'Mr Brown is at home at one time' may be true and that 'Mr Brown is at home at some other time' may be false.

§ 8. We propose now to consider the *Principles of Adjectival Determination* with a view to giving added significance to the predicational factor by bringing out the relation of an adjective to its determinable. For this purpose the principles will be reformulated as follows:

(1) *Principle of Implication*: If s is p, where p is a comparatively determinate adjective, then there must be some determinable, say P, to which p belongs, such that s is P.

(2) *Principle of Counterimplication*: If s is P, where P is a determinable, then s must be p, where p is an absolute determinate under P.

(3) *Principle of Disjunction*: s cannot be both p and p', where p and p' are any two different absolute determinates under P.

(4) *Principle of Alternation*: s must be either not-P; or p or p' or p''... continuing the alternants throughout the whole range of variation of which P is susceptible—p, p', p''... being comparatively determinate adjectives under P.

For convenience of reference, these formulae may be elliptically restated as follows:

(1) If s is p, then s is P.

(2) If s is P, then s is p.

(3) s cannot be both p and p', nor p' and p'', nor p and p''....

(4) s must be either not-P or p or p' or p'' or p'''....

Contrasting this reformulation with the original formulation of the principles of adjectival determination, it will be observed that, while the predications of s are more precise, they are not so palpably obvious. The force of the first principle is that, if a subject is of such a kind that a certain determinate adjective can be predicated of it, then this presupposes that the subject belongs to a certain category such that it may be compared in character with other subjects belonging to the same category, the ground of comparison being equivalent to the determinable. The second principle states that any subject whose character is so far known that a certain determinable adjective can be predicated of it, must in fact be characterised by some absolutely determinate value of that determinable, and that, although, in many cases, such a precise determination of character is impossible, yet the postulate that *in fact* the subject has some determinate character is one that reason seems to demand. The third principle as reformulated gains in significance, as compared with the mere disjunction of p with the indeterminate not-p, since now it precludes the possibility of conjoining an indefinite number of pairs of predicates, which are here exhibited as determinate and positive. In fact, in the principle of disjunction in its original form (according to which p cannot be joined with not-p) not-p should

signify—not merely some or any adjective *other than* p—but some adjective that is *necessarily incompatible* with p, and the only such adjectives are those other than p which belong to the same determinable.

The significance of the Principle of Alternation in its new form requires special discussion. It is developed from the dichotomy 'Any subject must be either not-P or P' where P stands for any determinable. This again assumes that 'Some subjects are not P,' i.e. that there are subjects belonging to such a category that the determinable adjective P is *not* predicable of them. The negative here must be termed a *pure* negative, in the sense that not-P cannot be resolved into an alternation of positive adjectives. For example, in the statement: 'Material bodies are not conscious' the negative term 'not conscious' does not stand for any single positive determinable which would generate a series of positive determinates. We ought in fact to maintain that 'not-conscious' is not properly speaking an adjective at all; for in accordance with the reformulated Principle of Adjectival Implication, every adjective that can be predicated of a subject must be a more or less determinate value of some determinable[1]. Eliminating, then, the negative not-P from the predicate, the reformulated Principle of Adjectival alternation may now be expressed in the form: 'Any subject that is P must be either p or p' or p'' or...' where the alter-

[1] A negative predication of this type has sometimes been called *Privative*; but unfortunately the term privative has also been used in an opposite sense, namely, for a predication applied to a subject belonging to a category for which the positive adjective is normally applicable; as when we predicate of a person that he is *blind* or that he is (temporarily) *unconscious*.

native predication p or p' or p'' or ... is restricted to subjects of which the determinable P is predicable. If we compare this form of the Principle of Alternation with the Principle of Counterimplication, viz., If s is characterised by P, it must be characterised by one or other determinate value of P, there would appear to be no obvious difference between them. The Principle of Alternation, however, supplements that of Counterimplication by implicitly postulating that the range of possible variation of the determinable can be apprehended in its completeness. The question whether we *can* in this way apprehend the complete range of possible variation of any determinable must be examined in detail. Consider the determinable 'integral number' which is always predicable of a collection or aggregate as such. Of a 'collection' we can in the first place assert universally that 'it is either zero or greater than zero,' and this it is to be observed, goes beyond the mere assertion that it is 'either zero or *not* zero.' Again we may assert that any collection is 'either zero or one or more than one,' where the alternant 'more than one' is not merely negative, but positive—though comparatively indeterminate. Proceeding in this way, we may resolve exhaustively the range of possible variations of number by an enumerated and finite series of positively indicated alternants: 'zero or one or two or three or...or n or greater than n.' What is here said of integral number holds of quantity in general, and may also be applied to any determinable (continuous or discrete) whose determinates have an order of betweenness and can therefore be serially arranged. For example, the range of hue

can be exhaustively resolved into the nine alternants 'red, or between red and yellow, or yellow, or between yellow and green, or green, or between green and blue, or blue, or between blue and violet, or violet.'

We may summarise (with some additional comments) what has been said with respect to the Principles of Adjectival Determination, formulated with reference to the determinable. (1) If s is p, then s is P. This postulates that whenever a comparatively determinate predication is asserted, then a determinable to which the determinate belongs can always be found; but it must be pointed out that language does not always supply us with a *name* for the determinable. (2) If s is P, then s is p. This postulates that in actual fact every adjective is manifested as an absolute determinate; it is to be supplemented, however, by the recognition that for a *continuously* variable determinable it is impossible actually to characterise a given subject by a precisely determinate adjective. (3) s cannot be both p and p'. This asserts that any two different determinates are incompatible; but, inasmuch as we are unable practically to characterise an object determinately (in the case of a continuously variable determinable), we must apply the formula to the case where p and p' (though only comparatively determinate) are figuratively speaking 'outside one another.' To represent this figurative analogy, suppose a point (a, b, c or d) to represent an absolute determinate, and the segment of a line (p or p') to represent a comparative determinate:

then, if b is between a and c, and only then, can we assert that the (comparative) determinates p and p' are codisjunct or incompatible. (4) Any s that is P must be either p or p' or.... Here the predications p, p', p'', etc. need not be absolute determinates, but to render the principle practically significant it is necessary that we should be able to compass in thought the *entire* stretch or range of variation of which P is susceptible.

§ 9. We will now pass to the principles according to which the manifested value of any one variable is determined by *its connection* with the manifested values of other variables. These principles may be expressed in forms analogous to those of adjectival determination and will be entitled the *Principles of Connectional Determination.* They embody the purely logical properties of the causal relation; but the notion of cause and effect—being properly restricted to phenomena temporally alterable—will be replaced by the wider notion of determining and determined. The characters which may be said jointly to *determine* some other character correspond to what is commonly called the cause, while any character which is thereby determined corresponds to an effect. Analysis of the general conception of causal connection reveals two complementary aspects which may be thus expressed: (a) wherever, in two instances, there is *complete agreement* as regards the cause-factors, there will be agreement as regards any effect-factor; and (b) wherever, in two instances, there is any (partial) difference as regards the cause-factors, there will be *some* difference in one or other of the effect-factors. In formulating the Principles of Con-

nectional Determination, such symbols as P, Q, R, T, will be introduced to represent the characters that are connectionally *determined*, along with A, B, C, D, to represent those which connectionally determine the former. Thus the conjunction $abcd$ would correspond to a cause-complex, and $pqrt$ to an effect-complex.

Principles of Connectional Determination

(1) *Principle of Implication.* Taking any determinable P, the determinate value which it assumes in any manifestation is determined by the conjunction of a finite number of determinables A, B, C, D (say), such that any manifestation that has the determinate character $abcd$ (say) will have the determinate character p (say).

(2) *Principle of Counterimplication.* Taking any determinable A, the determinate value which it assumes in any manifestation determines (in conjunction with other factors) a conjunction of a finite number of determinables P, Q, R, T (say), such that if, for instance, some manifestation having the determining character a has the determined character $pqrt$, then any manifestation that has the (different) character a' will have one or other of the different characters p' or q' or r' or t' (say).

(3) *Principle of Disjunction.* P being one of the characters determined by the conjunction of the determining characters A, B, C, D, there can be no three instances characterised respectively by

$$abcd \sim p, \quad a'bcd \sim p', \quad a''bcd \sim p.$$

(4) *Principle of Alternation.* On the same hypothesis, it must be that either 'every $abcd$ is p' or

'every *abcd* is p'' or 'every *abcd* is p''' or, etc., where the range of alternation covers all possible determinate values of P.

The Principle of Implication postulates that the determinate value assumed by any variable is dependent in any instance not upon an indefinite number of conditions which might in some sense be exhaustive of the whole state of the universe, but upon a set of conditions that are capable of enumeration. The theoretical and practical possibility of enumerating the factors which together constitute the determining complex, enables us to express the nature of reality in universal propositions of the form 'every *abcd* is p.' If the character p could be predicated universally only of a class determined by an infinite number of conjoined characters, reality could not be described by means of universal propositions; or in other words, nature would not present uniformities which could be comprehended by thought; in short there would be nothing that could be called Laws of Nature. Hence the significance of our first principle is that reality presents uniformities that can be comprehended in thought, and that, whatever variable aspect of the universe we may be concerned with, a uniformity or law could be found such that from it the value of the variable in any manifestation could be inferred from knowledge (at least theoretically possible) of the values assumed by other variables. The Principle of Implication represents that more familiar aspect of the so-called Law of Causation expressed in terms of agreement: that in any two instances where there is complete agreement as regards the cause complex, there will be agreement

as regards the effect; or, still more colloquially, the same cause entails the same effect.

Turning now to the Principle of Counterimplication, this represents the other and complementary aspect of causation; namely that of difference. It postulates that we can by enumeration exhaust the characters that are determined in their variation by any cause complex; just as we assumed that the cause complex in the previous principle could be exhaustively described. In other words the effect, determined by any variation in the causal or determining complex, does not permeate the whole universe, but is restricted to some assignable sphere. This important postulate being presumed, the principle proceeds to state that, if any variable presents a different value in two instances, indications of this difference will be shown in one or other of the variables that are affected or determined by the given variable. This principle is therefore complementary to the preceding one; whereas the Principle of Implication asserts that where there is agreement in the cause there will be agreement in the effect, the Principle of Counterimplication asserts that where there is difference in the cause there will be a difference in the effect. It may perhaps even be said that in the popular conception of cause this latter aspect—viz. of difference—is more prominent than the former, viz. agreement. Here we must point out that the principles are not parallel, inasmuch as *complete agreement* in the cause is required to ensure agreement in the effect, whereas any *partial* difference in the cause will entail some difference in the effect.

A word must be said about the strictly formal relations between these Principles of Implication and Counterim-

plication. By what is familiarly known as inference by contraposition, the proposition 'Every *abcd* is *p*' is equivalent to the proposition 'Every p' is either a' or b' or c' or d'.' Similarly the proposition 'Every a' is p' or q' or r' or t'' is equivalent to 'Every *pqrt* is *a*.' Applying this formal contraposition to the formulae for cause and effect, we see that the proposition that 'The same cause always entails the same effect' is logically equivalent to 'Any difference in the effect would entail some difference in the cause'; and again the proposition that 'Any difference in the cause will entail some difference in the effect' is logically equivalent to 'The same effect always entails the same cause.' It will be thus seen that the implicative and counterimplicative principles are not obtainable one from the other as equivalents by contraposition, but are complementary to one another, so that taken together they represent the relation between cause and effect as reciprocal. Take the one aspect of this relation; then plurality of cause holds in the sense that the effect may be partially the same in two instances where the cause is different; and plurality of effects holds in the same sense, namely, that the cause may be partially the same in two instances where the effect is different. Take the other aspect of the relation: thus, when the effect is completely and determinately characterised the character of the cause is thereby uniquely determined, just as when the cause is completely and determinately characterised the character of the effect is thereby uniquely determined. Thus, whether we are considering the relation of cause to effect or of effect to cause, the principles postulated will be in terms of complete agreement or of partial difference.

We pass now to the Disjunctive Principle. In order to expound this we must consider three instances of *ABCD* which agree as regards the determinate values of all but one, viz. *A*, of these determinables. Then, taking into consideration the Counterimplicative Principle, a difference as regards *A* in two instances would entail some difference in one or other of the characters that are determined by the complex *ABCD*. The principle then states that, if, in some pair of instances, a variation in the determining factor *A* entails a variation in the selected character *P*, then any further variation in *A* would entail a further variation in this same character *P*; whereas if, in two instances, a variation in *A* entails no variation in *P*, then any further variation in *A* would entail no further variation in the same character *P*. It is essential to note that the Disjunctive Principle could not have been formulated as a disjunction of two types of instance, such as $abcd \sim p$ and $a''bcd \sim p$. This disjunction would be equivalent to asserting that a variation in any determining factor such as *A* would entail a variation in *any or every* determined factor such as *P*; whereas the Counterimplicative Principle has laid down only that a variation in *A* would entail a variation in *one or other* of the determined characters and not necessarily in every one of them. The Principle of Disjunction then supplements that of Counterimplication by maintaining that if *some one* variation in *A* entails a variation in the selected character *P*, then *any* variation in *A* would entail a variation in the same character *P*. It might be supposed, in the case where a variation of *A* entails no variation in *P*, that *P* is not causally connected with *A*, and that therefore *A* could be elimi-

nated. But the mere elimination of A is not in general permissible, since the character P in some one of its determinate values requires that A should be manifested in some or other of its determinate values; though, as regards the determinate value of P, it may be a matter of indifference what specific value A has. Since in this case A cannot be eliminated, it would be symbolically requisite to express the relation of determination for the case under consideration—not in the form '*bcd* determines p'—but in the form '*Abcd* determines p,' where the significance of the symbol A is that *any* determinate value may from instance to instance be manifested without affecting the determinate value p.

Fourthly, the Alternative Principle of Connectional Determination, asserts an alternation of universal propositions, and of course goes beyond any statement that could be derived from the Principle of Adjectival Alternation, in which the alternative is in the predicate. Thus the latter states the universal proposition that 'Every *abcd* is p or p' or p'' or ...' whereas the principle under present consideration states an alternation between the universal propositions 'Every *abcd* is p' or 'Every *abcd* is p'' or ....

These principles will be very much more fully discussed when we deal with the topic of formal or demonstrative induction; they have been introduced at this early stage of our logical exposition in order to indicate the nature of the transition from the Principles of Propositional Determination which are purely axiomatic, to those of Adjectival Determination under a determinable, which have the character partly of axioms and partly of postulates, and from these again to the Principles of

Connectional Determination which may be taken as pure postulates.

§ 10. The formulation of the principles of connectional determination has an important bearing upon the problem of internal and external relations. In controversies on this topic it appears to be agreed that the division of relations into internal and external is both exclusive and exhaustive; and yet there seems to be no agreement as to what precisely the distinction is. One school holds that all relations are internal; the other that all are external. But on the face of it it would appear that some must be internal, others external; for otherwise it would seem impossible to give meaning to the distinction. It will be found, however, that those who deny external relations doubt, for instance, not whether spatial and temporal relations are properly to be called external, but rather whether space and time are themselves real in the sense that the real can be truly *characterised* by spatial and temporal relations; those on the other side who deny internal relations apparently hold that the independent otherness of the terms of the relation renders the relation external, inasmuch as the specific and variable relation of one term to another is not that which determines or is determined by the mere existence of the one or of the other term. The adherents then of the exclusively internal view of relations hold that the relation and its terms are mutually determinative, and the adherents of the exclusively external view, that the relation and its terms are mutually non-determinative or independent. Now it appears to me that the root misunderstanding amongst the two schools of philosophy on this point is,

not as to what is meant by an internal as contrasted with an external relation, but rather what is the nature of the terms between which the relation is supposed to subsist. The one school maintains that the relation subsists between the characters of the two related terms; the other that it subsists between the terms themselves. According to the former contention, relations are internal in the sense that they depend wholly upon the character of the terms related; according to the latter, they are external in the sense that they do not depend at all upon the mere existence of the terms qua existents. In this connection there is a further source of confusion, namely as to whether in the character of a term are to be included such relations as those of space and time, these being admittedly external, in contrast to qualities proper which are admittedly internal.

At this point I will state my solution of the problem, which will appear so simple that it would seem difficult to account for the origin of the controversy. I hold, then, that relations between adjectives as such are internal; and those between existents as such are external. In this account, adjectives are to include so-called external relations, even the characterising relation itself, as well as every other relation. The otherness which distinguishes the 'this' from the 'that' is the primary and literally the sole external relation, being itself direct and underived. And this relation is involved in every external relation. In fact, qua existent, the 'this' and the 'that' have no specific relation. The specific external relations that hold of one to another existent are derivative from their characters, in the

wider sense of character. Thus, the relation of the 'this' to the 'that' obtained from the fact that 'this is blue and that is green,' is derived from the nature of the qualities blue and green. Again the relation of proximity or remoteness obtained from the fact that 'this is here and that is there,' is derived from the positions of the 'this' and the 'that' by which their specific spatiality is characterised. The most important application of the distinction is to causal and other forms of connectional determination. Here the primary relation called cause is that between the character, dating, and locating, of two occurrences, from which the relation between the occurrences themselves is derived, the former being internal and the latter external. If there were no such internal causal relation, nothing could be stated as to the relation of event to event, except that the one is invariably accompanied by the other in a certain assignable spatial relation of space and time; and even this external relation is derived from the internal relation subsisting between the temporal and spatial *positions* occupied by the two events. If, however, all spatial properties were relative, as is maintained by Einstein and his followers, there would be no spatial relations other than internal, in fact nothing to distinguish a space from that which occupies it. The principles of connectional determination have therefore been expressed directly in terms of the characters by which the manifestations of reality may be described, from which must be derived the external relations between such manifestations themselves. It will have been observed that the correlative notions of determination and dependence enter into the formulation

of the principles as directly applicable to the characters of manifestations and therefore only derivatively to the manifestations themselves. Hence the potential range for which these principles hold extends beyond the actually existent into the domain of the possibly existent. In this way the universality of law is wider than that of fact. While the universals of fact are implied by the universals of law, the statement of the latter has intrinsic significance not involved in that of the former.

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